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# **RGAP**: Output Gap Estimation in `R` *

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**Abstract**

Assessing potential output and the output gap is essential for policy-making and fiscal surveillance. The European Commission proposes a production function methodology that involves the estimation of two classes of Gaussian state space models. This paper presents the `R` package **RGAP** which features a flexible modeling framework for the appropriate bivariate unobserved component models and offers frequentist as well as Bayesian estimation techniques. Additional functionalities include direct access to the AMECO database and automated model selection procedures. Multiple illustrative examples outline data preparation, model specification, and estimation processes using **RGAP**.

**JEL classification**: C11, C32, C87, E24, E31, E32, E37, E62.
**Keywords**: business cycle, output gap, potential output, state space models, Kalman filter and smoother, Gibbs sampling

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# 1 Introduction

Monitoring potential output and its difference to actual output – the output gap – is vital for economic policy-making and fiscal surveillance.[1] While potential output quantifies the level of sustainable and non-inflationary output, the output gap measures short-term fluctuations from it (Hall and Taylor, 1991). A positive output gap indicates an overheating of the economy imposing inflationary pressure, a negative one suggests an economic downturn.

The estimation of potential output and the output gap is a challenge, since both quantities are unobservable. Many univariate as well as multivariate filtering techniques have been proposed (Watson, 1986; Laxton and Tetlow, 1992; Kuttner, 1994; Hodrick and Prescott, 1997; Blagrave et al., 2015). Since 2002, the European Commission (EC) has been using a Cobb-Douglas production function methodology to estimate the output gap for its member states (Havik et al., 2014). Potential output is split up into three components: non-financial capital stock, trend labor input, and the trend of total factor productivity (TFP). Total factor productivity represents the part of output that cannot be explained by the quantitative use of the production factors labor and capital and includes both the efficiency level and the degree of utilization of these two production factors. The potential labor supply is defined as the level of labor input consistent with the non-accelerating wage rate of unemployment (NAWRU), i.e., the level of unemployment at which increases in wage inflation do not accelerate inflation.

In this production function framework, computation of potential output essentially boils down to estimating the TFP trend and the NAWRU by applying the Kalman filter to two bivariate state space models. Since the resulting estimates in turn play a major role in setting budgetary constraints, transparency and reliability of the estimation techniques and results are essential. For the computation of the output gap, the EC offers the

[1]Under the Treaty on Stability, Coordination and Governance, member countries of the European Union (EU) are obliged to maintain a balanced budget, i.e., general and structural budget deficits shall not exceed certain thresholds. An estimate of the output gap is used to extract the structural budget deficit.

program **GAP**, which consists of an Excel interface, a Matlab-based graphical interface for Bayesian estimation methods, and a Fortran program which operates all routines. In addition, a similar implementation in Python has been published by the EC in 2021 (Blondeau et al., 2021). Compared to **GAP**, the Excel user interface has been simplified, enabling the computation of the NAWRU, the TFP trend and finally the output gap in one Excel file.

**RGAP** aims at complementing the existing tools for the estimation of the output gap using the EC production function methodology. The package is user-friendly and offers flexible model specifications to cope with differences across countries. **RGAP** provides both Maximum Likelihood Estimation (MLE) as well as a Bayesian Markov chain Monte Carlo (MCMC) procedures to obtain estimates of the latent state variables and the involved model parameters in the EC methodology. In addition, the package offers two alternative estimation methods: A model-based approach suggested by Kuttner (1994) and a univariate filtering procedure (Hodrick and Prescott, 1997). Finally, this documenation provides a comprehensive and rigorous overview of the methodology and the applied estimation techniques, and it serves as a detailed guide for package users. The main difference to the existing tools by the EC are the user operability options. While the EC tools rely on graphical user interfaces in Excel, the `R` package **RGAP** is used within `R` and focuses on user-friendliness, flexibility and integratability into existing routines. The package offers direct access to the AMECO database, emphasizing its easy applicability. **RGAP** is suitable for both beginners and advanced `R` users. If desired, the package can automatically select and estimate suitable models based on the data.

Section 2 discusses the production function methodology and two involved state space model families. The two alternative approaches are presented in Section 3. Section 4 describes basic theory regarding Gaussian state space modeling, Kalman filtering and smoothing, constrained MLE, and gives a brief derivation of the posterior distributions for the Bayesian approach. In Section 5, the general functionality of **RGAP** is outlined and Section 6 illustrates its usage by applying it to different European countries. Section

7 makes a short comparison between **RGAP** and the EC software **GAP** and Section 8 concludes.

## 2 The production function methodology

In this section, we outline the production function methodology to compute the output gap, with particular emphasis on the method employed by the European Commission (Havik et al., 2014). In some instances, we deviate from the EC method to allow for useful generalizations.

The output gap is defined as the percentage difference between realized output $y_t$ and potential or trend output $\overline{y}_t$, i.e.,

$$gap_t := \frac{y_t}{\overline{y}_t} - 1.$$

A positive output gap indicates that an economy is performing above its potential, while a negative output gap signals a downturn, possibly resulting in deflationary pressure.

Let $l_t$ be labor input and $k_t$ capital input. By $(cu_l, cu_k)_t$ we denote the degree of capacity utilization and by $(e_l, e_k)_t$ the degree of efficiency for both labor and capital. Output is modeled via a Cobb-Douglas production function, i.e.,

$$\begin{aligned} y_t &= l_t^{\alpha} \cdot k_t^{1-\alpha} \cdot tfp_t \qquad (1) \\ &= (l \cdot cu_l \cdot e_l)_t^{\alpha} \, (k \cdot cu_k \cdot e_k)_t^{1-\alpha} \, , \end{aligned}$$

where $tfp_t = (cu_l \cdot e_l)_t^{\alpha} \, (cu_k \cdot e_k)_t^{1-\alpha}$ denotes total factor productivity and $\alpha$ and $1 - \alpha$ represent the output elasticity of labor and capital, respectively.[2] Potential output $\overline{y}_t$ is defined as the trend level of output, i.e.,

$$\overline{y}_t = \left(\bar{l}_t\right)^{\alpha} \cdot (k_t)^{1-\alpha} \cdot \overline{tfp}_t,$$

where henceforth the bar indicates the respective trends series. Since capital stock enters without transformation, i.e. $k_t = \overline{k}_t$, the estimation of potential output essentially

[2] The EC suggests the usage of $\alpha = 0.65$ for all member countries. Under the assumption of competitive markets these represent the labor share in national income.

breaks down to removing the cyclical component from both labor input and total factor productivity.[3]

## 2.1 Trend labor

Labor input is modeled as the total number of hours worked which can be decomposed into

$$l_t = \left( popw_t \cdot parts_t \cdot \left(1 - \frac{ur_t}{100}\right) + lfnd_t \right) \cdot ahours_t,$$

where $popw_t$ denotes the population of working age, $parts_t$ the labor participation rate, $ur_t$ the unemployment rate, and $ahours_t$ the average number of hours worked. The domestic labor force is given by $popw_t \cdot parts_t$ whereas $lfnd_t$ represents workers without residency, such as cross-border commuters.[4] Labor input at the trend level is given by

$$\bar{l}_t = \left( popw_t \cdot \overline{parts}_t \cdot \left(1 - \frac{nawru_t}{100}\right) + \overline{lfnd}_t \right) \cdot \overline{ahours}_t,$$

where the $nawru_t$ denotes the non-accelerating wage rate of unemployment, i.e., the level of unemployment at which increases in wages do not accelerate inflation.

All trend components except for the $nawru_t$ are computed using the Hodrick-Prescott (HP) filter (Hodrick and Prescott, 1997). The $nawru_t$ is estimated using an unobserved component model. Thereby, the unemployment rate splits up into the sum of a trend component $p_t$ and a cycle $c_t$, i.e.,

$$ur_t = p_t + c_t. \tag{2}$$

Given the time series properties of the unemployment rate, the EC proposes to model the trend component $p_t$ as a local linear trend model.[5] The cycle component $c_t$ is modeled as an autoregressive (AR) process of order two. The supplementary equations are given

[3]Potential factor use of capital can simply be seen as the full utilization of existing capital stock in the economy (Havik et al., 2014). In addition, the contribution of capital to potential output is relatively stable, as the fraction of new investments to existing capital stock is small.

[4]Originally, the EC does not distinguish between residents and non-residents. In many economies, the effect of the part of the labor force without residency is negligible since the balance between outflow to and inflow from neighboring countries is small. Seeing that particularly small open economies might rely heavily on the inflow of labor, we include this feature.

[5]Local linear trend models are sometimes referred to as second order random walks.

by

$$\begin{aligned} p_t &= p_{t-1} + \eta_{t-1} + \varepsilon_{p,t}, & \varepsilon_{p,t} &\sim \mathcal{N}\left(0, \sigma_p^2\right) \\ \eta_t &= \eta_{t-1} + \varepsilon_{\eta,t}, & \varepsilon_{\eta,t} &\sim \mathcal{N}\left(0, \sigma_\eta^2\right) \\ c_t &= \phi_{c,1} c_{t-1} + \phi_{c,2} c_{t-2} + \varepsilon_{c,t}, & \varepsilon_{c,t} &\sim \mathcal{N}\left(0, \sigma_c^2\right) \end{aligned} \tag{3}$$

where $\varepsilon_{p,t}, \varepsilon_{\eta,t}, \varepsilon_{c,t}$ are independent. Some variations are possible, e.g., the trend could be modeled as random walk with constant drift, setting $\varepsilon_{\eta,t} = 0$ for all $t$. Moreover, the cycle could instead be modeled as an AR(1) process.

To facilitate the identification of the unobserved components, the cycle – or unemployment gap – is captured through a Phillips curve equation. The EC proposes two general specifications for the Phillips curve, a traditional one and a new Keynesian one. Under the traditional approach, the indicator builds on nominal unit labor costs $nulc_t$ defined by the compensation per employee relative to labor productivity. The relationship may include additional exogenous variables $z_t$. The backward-looking Philliüs curve is given by

$$\Delta^2 \log nulc_t = \mu + \beta_{c,0} c_t + \vartheta' z_t + \varepsilon_{nulc,t}, \qquad \varepsilon_{nulc,t} \sim \mathcal{N}\left(0, \sigma_{nulc}^2\right) \tag{4}$$

where $\vartheta$ is a vector of the same size as $z_t$. Possible modifications include additional cycle lags.

Under the New-Keynesian approach, the traditional relationship is altered to include forward-looking behavior in wage setters. This hybrid version of the Phillips curve considers real unit labor costs $rulc_t$ as wage indicator, i.e., $nulc_t$ is additionally adjusted for the price level. Contrary to above, the equation does not include additional exogenous variables. Depending on the autoregressive structure of the cycle term, the forward-solution of the Phillips curve implies parameter restriction on the the first cycle lag. Given an AR(2) cycle, we have

$$\begin{aligned} \Delta \log rulc_t &= \mu + \xi \Delta \log rulc_{t-1} + \beta_{c,0} c_t + \delta \beta_{c,0} \phi_{c,2} c_{t-1} + \varepsilon_{nulc,t}, \\ &\varepsilon_{nulc,t} \sim \mathcal{N}\left(0, \sigma_{nulc}^2\right), \end{aligned} \tag{5}$$

where $\delta$ denotes the consumption discount factor and $\xi$ measures the degree of backwardness.[6] For an AR(1) cycle, the cycle lag drops from the equation. For a derivation of the forward-solution shown in Equation (5), see Appendix A.1.

Havik et al. (2014) propose to stabilize the estimation of the NAWRU across vintages by setting an anchor for the medium to long term. The EC estimates the anchors in a panel regression across European countries using several labor market variables determining structural developments. The implementation of the anchor in the estimation of the NAWRU is detailed in Section 4.2.

## 2.2 Trend of total factor productivity

Total factor productivity is obtained as the residual of Equation (1). Also using Equation (1), we have that

$$\log tfp_t = \alpha \log \left(cu_l e_l\right)_t + (1-\alpha) \log \left(cu_k e_k\right)_t . \tag{6}$$

The degree of capacity utilization $(cu_l, cu_k)$ is measured by the aggregate indicator capacity utilization business sentiment (CUBS) which contains both a trend and a cyclical component (Havik et al., 2014). The goal is to estimate the efficiency terms $(e_l, e_k)$ which do not contain cyclical movements. Hence, Equation (6) can be simplified, i.e.,

$$\log tfp_t = p_t + c_t, \tag{7}$$

where $p_t := \alpha \log e_{l,t} + (1-\alpha) e_{k,t}$ is the unobserved permanent component and $c_t := \alpha \log cu_{l,t} + (1-\alpha) \log cu_{k,t}$ is the cyclical one. A key assumption is that $cubs_t$ and $cu_{k,t}$ are highly correlated (i.e., $cubs_t \approx cu_{k,t}$) and that there exists a correlation between $cu_k$ and $cu_l$ of the form

$$cu_{l,t} = \gamma cu_{k,t} + \varepsilon_t,$$

with $0 < \gamma < 1$. Then,

$$cubs_t \approx \frac{1}{1-\alpha+\alpha\gamma} c_t + \alpha \varepsilon_t \tag{8}$$

with $1/_{1-\alpha+\alpha\gamma} > 1$.

[6] The discount factor $\delta$ is set to 0.99 (see e.g. Hristov et al., 2017).

In order to extract the trend component $p_t$ in Equation (7), the relationship in Equation (8) can be used to estimate a second bivariate unobserved component model. The EC proposes to model the trend component as a damped drift and the cycle as a re-parameterized AR(2) process, i.e.,

$$
\begin{aligned}
p_t &= p_{t-1} + \eta_{t-1} + \varepsilon_{p,t}, & \varepsilon_{p,t} &\sim \mathcal{N}\left(0, \sigma_p^2\right) \\
\eta_t &= (1-\phi_\eta)\omega + \phi_\eta \eta_{t-1} + \varepsilon_{\eta,t}, & \varepsilon_{\eta,t} &\sim \mathcal{N}\left(0, \sigma_\eta^2\right) \\
c_t &= \phi_{c,1} c_{t-1} + \phi_{c,2} c_{t-2} + \varepsilon_{c,t}, & \varepsilon_{c,t} &\sim \mathcal{N}\left(0, \sigma_c^2\right)
\end{aligned}
\tag{9}
$$

where $\phi_{c,1} = 2A\cos\left({}^{2\pi}/_{\tau}\right)$ and $\phi_{c,2} = -A^2$ with mean cycle periodicity $\tau$ and amplitude $A$. This re-parametrization imposes complex conjugate roots and thereby implies that the autocorrelation function is an exponentially damped oscillation.

Building on Equation (8), the cyclical component is linked to the capacity utilization economic sentiment indicator $cubs_t$ via a second measurement equation

$$
\begin{aligned}
cubs_t &= \mu + \sum_{j=1}^{\tilde{p}} \phi_{cubs,j} cubs_{t-j} + \sum_{j=0}^{p} \beta_{c,j} c_{t-j} + \hat{\varepsilon}_t, \\
\hat{\varepsilon}_t &= \sum_{j=1}^{i} \phi_{\hat{\varepsilon},j} \hat{\varepsilon}_{t-j} + \varepsilon_{cubs,t}, \qquad \varepsilon_{cubs,t} \sim \mathcal{N}\left(0, \sigma_{cubs}^2\right).
\end{aligned}
\tag{10}
$$

The baseline model suggested by the EC consists of Equations (7) - (10) with $\tilde{p}, p, i = 0$ and $\beta_{c,0} > 1$ as implied by Equation (8). Yet, certain model variations are possible. For instance, the cycle may be modeled as an ordinary AR(1) or AR(2) process and the trend as a local linear trend or a random walk with drift. Moreover, for quarterly data, additional cycle or autoregressive lags in the CUBS equation might improve the model fit.

# 3 Alternative frameworks

Since potential output and the output gap are unobservable, an empirical validation of the obtained estimates is not possible and their interpretation demands caution. A comparison to other modeling frameworks is helpful in assessing the certainty of the estimates. We consider two popular alternative methodologies for the estimation of potential output and the output gap. The first one is the standard HP-filter (Hodrick and Prescott, 1997), which is a univariate filtering procedure based solely on observed output, thereby ignoring economic links between output, inflation and unemployment. The second one is the bivariate unobserved component model by Kuttner (1994). In contrast, the latter model connects deviations of output from its potential with inflation. To be more precise, potential output is defined as the level of output for which inflation is constant.

Let $y_t$ denote real output and let $\pi_t$ be a measure of inflation. Then, Kuttner's model is given by

$$
\begin{aligned}
\log y_t &= p_t + c_t \\
p_t &= \mu_p + p_{t-1} + \varepsilon_{p,t}, & \varepsilon_{p,t} &\sim \mathcal{N}\left(0, \sigma_p^2\right) \\
c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_{c,t}, & \varepsilon_{c,t} &\sim \mathcal{N}\left(0, \sigma_c^2\right) \\
\Delta\pi_t &= \mu_\pi + \gamma \Delta \log y_{t-1} + \beta_{c,0} c_{t-1} + \varepsilon_{\pi,t} \\
&\quad + \delta_1 \varepsilon_{\pi,t-1} + \delta_2 \varepsilon_{\pi,t-2} + \delta_3 \varepsilon_{\pi,t-3}, & \varepsilon_{\pi,t} &\sim \mathcal{N}\left(0, \sigma_\pi^2\right)
\end{aligned}
$$

where $p_t$ and $c_t$ again denote the trend and cycle components, respectively. The cycle is modeled as an AR(2) process and the trend as a random walk with drift. The inflation equation features a third order moving average innovation process, lagged output growth, and the lagged cycle component. Analogously to the TFP and NAWRU estimation of the production function methodology, the Kuttner model can be estimated using the Kalman filter and maximum likelihood estimation.

# 4 Gaussian state space models and estimation

The unobserved component models outlined in the previous section can be cast into a bivariate state space representation and subsequently estimated using the Kalman filter. **RGAP** provides simple tools to define the appropriate state space models within the production function framework. It thereby uses the extensive state space modeling implemented in **KFAS** (Helske, 2017). In this section, we briefly summarize key aspects of

Gaussian state space modeling, mostly adopting the notation used by Durbin and Koopman (2012) on which **KFAS** to a large extent builds. In addition, we formulate state space representations for certain specifications of the NAWRU and TFP model.

We consider the linear Gaussian state space model

$$\begin{aligned} y_t &= Z_t \alpha_t + \varepsilon_{y,t}, \qquad \varepsilon_{y,t} \sim \text{ i.i.d } \mathcal{N}\left(0, H_t\right), \\ \alpha_{t+1} &= T_t \alpha_t + R_t \varepsilon_{\alpha,t}, \quad \varepsilon_{\alpha,t} \sim \text{ i.i.d } \mathcal{N}\left(0, Q_t\right), \end{aligned} \tag{11}$$

for $t = 1, \ldots, n$ with $\mathbb{E}\left[\varepsilon_{y,t}\varepsilon'_{\alpha,t}\right] = 0$, $\alpha_1 \sim \mathcal{N}\left(a_1, P_1\right)$ and where $y_t$ is the $p \times 1$ observation vector and $\alpha_t$ is the unobserved $m \times 1$ state vector. The dimensions of all vectors and matrices are given in Table 1. We call the first equation of (11) observation equation and the second equation is called state equation. The $m \times r$ matrix $R_t$ is included to ensure that the covariance matrix $Q_t$ is positive definite. $R_t$ is usually a subset of the columns of $I_m$ and is therefore called selection matrix.

| State space model<br>Vector | | Matrix | | Kalman filter an smoother<br>Vector | | Matrix | |
|---|---|---|---|---|---|---|---|
| $y_t$ | $p \times 1$ | $Z_t$ | $p \times m$ | $a_t$ | $m \times 1$ | $P_t$ | $m \times m$ |
| $\alpha_t$ | $m \times 1$ | $T_t$ | $m \times m$ | $a_{t\|t}$ | $m \times 1$ | $P_{t\|t}$ | $m \times m$ |
| $\varepsilon_{y,t}$ | $p \times 1$ | $H_t$ | $p \times p$ | $\hat{\alpha}_t$ | $m \times 1$ | $V_t$ | $m \times m$ |
| $\varepsilon_{\alpha,t}$ | $r \times 1$ | $Q_t$ | $r \times r$ | $v_t$ | $p \times 1$ | $F_t$ | $p \times p$ |
| | | $R_t$ | $m \times r$ | | | $K_t$ | $m \times p$ |
| $a_1$ | $m \times 1$ | $P_1$ | $m \times m$ | | | | |

Table 1: Dimensions of the state space model in Equation (11) and of the Kalman filter and smoother recursions in Equations (12) – (14).

Note that for our purposes, the time-varying feature of the system matrices is in most cases obsolete. However, we incorporate exogenous variables to the observation equation via the time-varying matrix $Z_t$.

## 4.1 Kalman filtering and smoothing

Kalman filtering and smoothing is performed to obtain the unobserved variables stored in $\alpha$. Filtering gives the one-step-ahead predictions, prediction errors and their respective covariance matrices

$$a_{t+1} = \mathbb{E}\left[\alpha_{t+1} \middle| y^t\right], \qquad P_{t+1} = \text{var}\left(\alpha_{t+1} \middle| y^t\right),$$

$$v_t = y_t - Z_t a_t, \qquad\qquad F_t = \mathrm{var}\left(v_t \middle| y^{t-1}\right) = Z_t P_t Z_t' + H_t,$$

where by $y^t$ we denote the set of past information $\{y_1, \ldots, y_t\}$. Backward smoothing then yields

$$\hat{\alpha}_t = \mathbb{E}\left[\alpha_t | y^n\right], \qquad\qquad V_t = \mathrm{var}\left(\alpha_t | y^n\right).$$

Standard results on the multivariate Normal distribution yield the following Kalman recursion for the state space model in Equation (11).[7] The prediction step of the Kalman filter is given by

$$\begin{aligned} a_t &= T_t a_{t-1|t-1} + c, \quad & P_t &= T_t P_{t-1|t-1} T_t' + R_t Q_t R_t', \\ v_t &= y_t - Z_t a_t, & F_t &= Z_t P_t Z_t' + H_t, \end{aligned} \tag{12}$$

and the updating step by

$$\begin{aligned} a_{t|t} &= \mathbb{E}\left[\alpha_t \middle| y^t\right] = a_t + K_t v_t, \\ P_{t|t} &= \mathrm{var}\left(\alpha_t \middle| y^t\right) = P_t - K_t Z_t P_t, \end{aligned} \tag{13}$$

for $t = 1, \ldots, n$, where the matrix $K_t = P_t Z_t' F_t^{-1}$ is called Kalman gain. The distribution of the initial state $\alpha_t$ is assumed to be known. The fixed interval smoother based on the entire sample is given by the backward recursion

$$\begin{aligned} \hat{\alpha}_t &= \mathbb{E}[\alpha_t | y^n] = a_{t|t} + P_{t|t} T_t' P_{t+1}^{-1} (\hat{\alpha}_{t+1} - a_{t+1}), \\ V_t &= \mathrm{var}(\alpha_t | y^n) = P_{t|t} + P_{t|t} T_t' P_{t+1}^{-1} (V_{t+1} - P_{t+1}) P_{t+1}^{-1} T_t P_{t|t} \end{aligned} \tag{14}$$

for $t = n-1, \ldots, 1$ with $\hat{\alpha}_n = a_n$ and $V_n = P_n$. The dimensions of all vectors and matrixed involved in the Kalman filter and smoother equations (12), (13), and (14) are summarized in Table 1.

The Kalman filter recursions in Equations (12) and (13) are stated under the assumption that $a_1$ and $P_1$ are known. In practice, some or all of the elements of $a_1$ and $P_1$ are unknown, e.g. in the case where the state vector contains non-stationary elements. We therefore use the exact diffuse initialization method presented by Durbin and Koopman (2012) and de Jong and Mackinnon (1988) and implemented in **KFAS**.

[7] We refrain from giving a detailed derivation of the filtering and smoothing recursions and refer the interested reader to Durbin and Koopman (2012).

## 4.2 Conditional updating

The NAWRU model outlined in Section 2 is supplemented by an anchor $s_{n+h}$ at time $n+h$, where $h$ denotes the horizon beyond the end of sample $n$. Under the assumption that there is no policy change, the NAWRU is supposed to converge to the anchor representing structural unemployment. In an application to EU member states, Hristov et al. (2017) find that anchoring the NAWRU results in more moderate pro-cyclicality and in less pronounced revisions between estimations. The anchor itself is estimated via a panel regression of unemployment in EU countries on several structural labor market indicators (Orlandi, 2012).

In this section, we present the framework for updating the smoothed estimates of the NAWRU. In principle, this method can be applied to any component of the state vector, and is thus also available for the TFP and Kuttner model. We denote the state component to be anchored by $\hat{\alpha}_{i,t} = r_i \hat{\alpha}_t$. Similar to the selection matrix $R$, the $1 \times m$ selection vector $r_i$ contains a one in column $i$ and zeros elsewhere, loading the state variable. The anchored smoothed state variable is defined by $\hat{\alpha}^a_{i,t} = \mathbb{E}\left[\alpha_{i,t} | y^n, \alpha_{i,n+h} = s_{n+h}\right]$.

Given that the state space model we consider is Gaussian, it holds that

$$\begin{pmatrix} \hat{\alpha}_{i,t} \\ \hat{\alpha}_{i,n+h} \end{pmatrix} \Bigg| y^n \sim \mathcal{N}\left( \begin{pmatrix} \mathbb{E}\left[\alpha_{i,t} | y^n\right] \\ \mathbb{E}\left[\alpha_{i,n+h} | y^n\right] \end{pmatrix}, \Sigma \right)$$

where

$$\Sigma = \begin{pmatrix} \operatorname{var}\left(\alpha_{i,t} | y^n\right) & \operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h} | y^n\right) \\ \operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h} | y^n\right) & \operatorname{var}\left(\alpha_{i,n+h} | y^n\right) \end{pmatrix}.$$

Standard results on the multivariate Gaussian distribution imply that

$$\begin{aligned} \hat{\alpha}^a_{i,t} &= \mathbb{E}\left[\alpha_{i,t} | y^n, \alpha_{i,n+h} = s_{n+h}\right] \\ &= \mathbb{E}\left[\alpha_{i,t} | y^n\right] + \frac{\operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h} | y^n\right)}{\operatorname{var}\left(\alpha_{i,n+h} | y^n\right)} \left(s_{n+h} - \mathbb{E}\left[\alpha_{i,n+h} | y^n\right]\right). \end{aligned}$$

and thus, the anchor is equal to the smoothed estimate adjusted by the weighted difference between the anchor and the conditional forecast at horizon $h$ given $y^n$. Naturally, the weights $w_t := {\operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h} | y^n\right)}/{\operatorname{var}\left(\alpha_{i,n+h} | y^n\right)}$ decrease exponentially for $t \leq n+h$ and for

$t = n + h$, $w_t = 1$.

The weights $w_t$ and the conditional expectation $\mathbb{E}\left[\alpha_{i,n+h}|y^n\right]$ can be retrieved from the Kalman smoother. For $k \in \mathbb{Z}$ we have that

$$
\begin{aligned}
\mathbb{E}\left[\alpha_{i,n+k}|y^n\right] &= r_i \mathbb{E}\left[\alpha_{n+h}|y^n\right] = r_i T^k \mathbb{E}\left[\alpha_n|y^n\right] = r_i T^k \hat{\alpha}_n, \\
\operatorname{var}\left(\alpha_{i,n+k}|y^n\right) &= r_i \operatorname{var}\left(\alpha_{n+k}|y^n\right) r_i' \\
&= r_i \operatorname{var}\left(T\alpha_{n+k-1} + R_t \varepsilon_{\alpha,n+k-1}|y^n\right) r_i' \\
&= r_i \left(T \operatorname{var}\left(\alpha_{n+k-1}|y^n\right) T' + R_t Q R_t'\right) r_i' \\
&= r_i \left(T^k \operatorname{var}\left(\alpha_n|y^n\right) T^{k'} + \sum_{i=0}^{h-1} T^i R_t Q R_t' T^{i'}\right) r_i' \\
&= r_i \left(T^k P_{n|n} T^{k'} + \sum_{i=0}^{h-1} T^i R_t Q R_t' T^{i'}\right) r_i',
\end{aligned} \tag{15}
$$

where by convention $T^0 = I$, and similarly

$$
\begin{aligned}
\operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h}|y^n\right) &= r_i \operatorname{cov}\left(\alpha_t, \alpha_{n+h}|y^n\right) r_i' \\
&= r_i \operatorname{cov}\left(\alpha_t, T\alpha_{n+h-1} + R_t \varepsilon_{\alpha,n+h-1}|y^n\right) r_i'
\end{aligned}
$$

and thus

$$
\operatorname{cov}\left(\alpha_{i,t}, \alpha_{i,n+h}|y^n\right) = \begin{cases} r \operatorname{cov}\left(\alpha_t, \alpha_n|y^n\right) T^{h'} r' & \text{if } 1 \leq t < n, \\ r \operatorname{var}\left(\alpha_t|y^n\right) T^{h-t+n'} r' & \text{if } n \leq t \leq n+h. \end{cases}
$$

The second case we already know from Equation (15). For the first case, using the results of de Jong and Mackinnon (1988), we have that

$$
\operatorname{cov}\left(\alpha_t, \alpha_n|y^n\right) = \begin{cases} A_t \operatorname{cov}\left(\alpha_{t+1}, \alpha_n|y^n\right) & \text{if } t < n, \\ \operatorname{cov}\left(\alpha_t, \alpha_{n-1}|y^n\right) T_{n-1}' & \text{if } n \leq t, \end{cases}
$$

where $A_t = P_{t|t} T' P_{t+1}^{-1}$ for $t < n$. Hence, for $t < n$, $\operatorname{cov}\left(\alpha_{t+1}, \alpha_n|y^n\right)$ can be computed via a backward recursion with starting value $\operatorname{cov}\left(\alpha_n, \alpha_n|y^n\right) = \operatorname{var}\left(\alpha_n|y^n\right) = P_{n|n}$.

Summing up, the anchored state component $\hat{\alpha}_{i,t}^a$ is given by

$$
\hat{\alpha}_{i,t}^a = \hat{\alpha}_{i,t} + \frac{r A_t \operatorname{cov}\left(\alpha_{t+1}, \alpha_n|y^n\right) T^{h'} r'}{r \operatorname{var}\left(\alpha_{n+h}|y^n\right) r'} \left(s_{n+h} - r T^h \hat{\alpha}_n\right), \quad t < n,
$$

and

$$\hat{\alpha}^a_{i,t} = rT^{t-n}\hat{\alpha}_n + \frac{r\,\mathrm{cov}\left(\alpha_t, \alpha_{n+h}|y^n\right) r'}{r\,\mathrm{var}\left(\alpha_{n+h}|y^n\right) r'} \left(s_{n+h} - rT^h\hat{\alpha}_n\right), \quad n \leq t \leq n+h.$$

Detailed recursions are given in Appendix A.4.

## 4.3 Constrained MLE

The filtering equations in Equations (12) and (13) enable the specification of the likelihood function and the model parameters can be obtained by Maximum Likelihood Estimation (MLE). Using the estimated parameters and corresponding filtered states, applying Equation (14) gives the smoothed states.

The considered models involve several parameter constraints. The cycle component is assumed to be stationary and so is the trend drift in case of the damped trend model. While all system variances need to be positive, **RGAP** applies box constraints to ensure identifiable solutions. Moreover, the theoretical derivations of the Phillips curve as well as the CUBS equation imply additional parameter constraints (see Equation (8) and Appendix A.1).

In order to report the appropriate standard errors for the constrained optimization problem, we apply the delta method. For any function $g : \mathbb{R}^k \to \Theta$ that maps to the $k$-dimensional parameter space $\Theta$ and for which $\nabla g(\theta)$ exists,

$$\sqrt{n}\left(g\left(\hat{\theta}\right) - g(\theta)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \nabla g(\theta)' \Sigma \nabla g(\theta)\right)$$

(see e.g. Kim and Nelson, 1999). The fitting routines supplied by **RGAP** minimize the negative log-likelihood function, implying that the Hessian matrix returned by the optimization algorithm has a negative sign and thus the Fisher information matrix is equal to the returned Hessian. Hence, the covariance matrix $\Sigma$ can be approximated by the inverse of the Hessian.

The specific transformation functions alongside their derivatives can be found in Appendix A.2.

## 4.4 Bayesian estimation

As suggested by Havik et al. (2014), the model for total factor productivity can be estimated using Bayesian estimation methods. Informative priors allow for the inclusion of information based on macroeconomic theory, enabling a more precise and robust estimation compared to MLE. An additional advantages of applying Bayesian methods is the avoidance of the identification problem causing zero variance estimates. In what follows, we adopt and extend the propositions made in Havik et al. (2014). In addition, we provide prior distributions and methods to obtain their posterior for a general set of model specifications such that the NAWRU can also be estimated by Bayesian methods, which ist currently not possible with alternative software by the EC.

### 4.4.1 Prior distribution

To complete the Bayesian TFP and NAWRU models, it remains to specify prior distributions for the model parameters. Let $\theta_p, \theta_c$, and $\theta_{ind}, ind \in \{cubs, pci\}$ denote the set of parameters involved in the trend equation, cycle equation, and CUBS or Phillips curve indicator ($pci$) equation, respectively. Assuming a block independence structure, we have that

$$p(\theta) = p(\theta_p)\, p(\theta_c)\, p(\theta_{ind})\,.$$

For $\theta_{ind} = (\beta, \phi_{\hat{\varepsilon}}, \sigma^2_{ind})$ with $\beta = (\mu, \phi_{ind,1}, \ldots, \phi_{ind,\tilde{p}}, \beta_{c,0}, \ldots, \beta_{c,p})'$ and $\phi_{\hat{\varepsilon}} = (\phi_{\hat{\varepsilon},1}, \ldots, \phi_{\hat{\varepsilon},i})'$, we assume that

$$p(\theta_{ind}) = p\left(\beta, \sigma^2_{ind}\right) p(\phi_{\hat{\varepsilon}})$$
$$p\left(\beta, \sigma^2_{ind}\right) = \mathcal{NIG}\left(\beta_0, Q_{\beta 0}, s_{ind0}, \nu_{ind0}\right) I_{ind}$$
$$p(\phi_{\hat{\varepsilon}}) = \mathcal{N}\left(\phi_{\hat{\varepsilon}0}, V_{\phi_{\hat{\varepsilon}}0}\right) I_{\phi_{\hat{\varepsilon}}}$$

where $\mathcal{NIG}$ denotes the Normal-Inverse-Gamma distribution with precision $Q_{\beta 0}$ and the sets $I_{ind}$ and $I_{\phi_{\hat{\varepsilon}}}$ comprise the parameter and stationarity constraints.

For the trend model specification, we set $\sigma^2_p = 0$ which simplifies the analysis and additionally results in a relatively smooth trend. For convenience, we restate the first trend equation $p_t = p_{t-1} + \eta_{t-1}$ for which $\eta_t$ alongside all prior distributions is specified in Table 2.[8]

| Random walk with drift | Local linear trend | Damped trend |
|---|---|---|
| $\eta_t = \omega + \varepsilon_{\eta,t}$ | $\eta_t = \eta_{t-1} + \varepsilon_{\eta,t}$ | $\eta_t = (1-\phi_\eta)\,\omega + \phi_\eta \eta_{t-1} + \varepsilon_{\eta,t}$ |
| $\theta_p = \left(\omega, \sigma_\eta^2\right)$ | $\theta_p = \sigma_\eta^2$ | $\theta_p = \left(\omega, \phi_\eta \sigma_\eta^2\right)$ |
| $p(\theta_p) = p(\omega)\, p\left(\sigma_\eta^2\right)$ | $p(\theta_p) = p\left(\sigma_\eta^2\right)$ | $p(\theta_p) = p(\omega)\, p(\phi_\eta)\, p\left(\sigma_\eta^2\right)$ |
| $p\left(\sigma_\eta^2\right) = \mathcal{IG}\left(s_{\eta 0}, \nu_{\eta 0}\right) I_{\sigma_\eta^2}$ | $p\left(\sigma_\eta^2\right) = \mathcal{IG}\left(s_{\eta 0}, \nu_{\eta 0}\right) I_{\sigma_\eta^2}$ | $p\left(\sigma_\eta^2\right) = \mathcal{IG}\left(s_{\eta 0}, \nu_{\eta 0}\right) I_{\sigma_\eta^2}$ |
| $p(\omega) = \mathcal{N}\left(\omega_0, V_{\omega 0}\right) I_\omega$ | | $p(\omega) = \mathcal{N}\left(\omega_0, V_{\omega 0}\right) I_\omega$ |
| | | $p(\phi_\eta) = \mathcal{N}\left(\phi_{\eta 0}, V_{\phi_\eta 0}\right) I_{\phi_\eta}$ |

Table 2: Trend prior specifications. The first trend equation is given by $p_t = p_{t-1} + \eta_{t-1}$. For the error term we assume $\varepsilon_{\eta,t} \sim \mathcal{N}\left(0, \sigma_\eta^2\right)$. $\mathcal{IG}$ denotes the Inverse-Gamma distribution.

For the cycle equation, we consider three different specifications, namely, stationary autoregressive processes of order 1 and 2, and a re-parametrized AR(2) with complex conjugate roots. The specifications and respective prior distributions are given in Table 3.

| AR(1) | AR(2) | Re-parameterized AR(2) |
|---|---|---|
| $c_t = \phi_{c,1} c_{t-1} + \varepsilon_{c,t}$ | $c_t = \phi_{c,1} c_{t-1} + \phi_{c,2} c_{t-2} + \varepsilon_{c,t}$ | $c_t = 2A\cos\left(2\pi/\tau\right) c_{t-1} - A^2 c_{t-2} + \varepsilon_{c,t}$ |
| $\theta_c = \left(\phi_{c,1}, \sigma_c^2\right)$ | $\theta_c = \left(\phi_c, \sigma_c^2\right), \phi_c = (\phi_{c,1}, \phi_{c,2})$ | $\theta_c = \left(A, \tau, \sigma_c^2\right)$ |
| $p(\theta_c) = p(\phi_{c,1})\, p\left(\sigma_c^2\right)$ | $p(\theta_c) = p(\phi_c)\, p\left(\sigma_c^2\right)$ | $p(\theta_c) = p(A)\, p(\tau)$ |
| $p\left(\sigma_c^2\right) = \mathcal{IG}\left(s_{c0}, \nu_{c0}\right) I_{\sigma_c^2}$ | $p\left(\sigma_c^2\right) = \mathcal{IG}\left(s_{c0}, \nu_{c0}\right) I_{\sigma_c^2}$ | $p\left(\sigma_c^2\right) = \mathcal{IG}\left(s_{c0}, \nu_{c0}\right) I_{\sigma_c^2} p\left(\sigma_c^2\right)$ |
| $p(\phi_{c,1}) = \mathcal{N}\left(\phi_{c,1,0}, V_{\phi_{c,1}0}\right) I_{\phi_{c,1}}$ | $p(\phi_c) = \mathcal{N}\left(\phi_{c0}, V_{\phi_c 0}\right) I_{\phi_c}$ | $p(A) = \mathcal{B}\left(\alpha_{A0}, \beta_{A0}\right) I_A$ |
| | | $p(\tau) = \mathcal{B}\left(\alpha_{\tau 0}, \beta_{\tau 0}\right) I_\tau$ |

Table 3: Cycle prior specifications. For the error term we assume $\varepsilon_{c,t} \sim \mathcal{N}\left(0, \sigma_c^2\right)$. $\mathcal{IG}$ denotes the Inverse-Gamma distribution and $\mathcal{B}$ the beta distribution.

### 4.4.2 Posterior distribution

Given the model specification and the prior distributions, we now outline a procedure to obtain the posterior distribution of the parameters and the states.

Let now $y_t = (y_{1,t}, ind_t)'$, $t = 1, \ldots n$ be the observed data and define $y^n := (y_1, \ldots, y_n)$.[9] Analogously, for the state vector, $\alpha^n := (\alpha_1, \ldots, \alpha_n)$. The posterior distribution $p(\alpha^n, \theta | y^n)$

[8]For details on the parameterization of the Inverse-Gamma distribution, see Appendix A.5.4.
[9]In case of the TFP model, $y_{1,t} = \log tfp_t$, for the NAWRU model, $y_{1,t} = ur_t$.

is not given in closed form and thus, we use a Gibbs sampling procedure to obtain draws from it. For the posterior, it holds that

$$p(\alpha^n, \theta | y^n) = p(\alpha^n | \theta, y^n)\, p(\theta | \alpha^n, y^n)\,. \tag{16}$$

Samples from the first term on the right-hand side can be readily obtained by applying the Kalman filter and smoother conditional on the model parameters and the data (Carter and Kohn, 1994). To see that, we can factorize

$$p(\alpha^n | \theta, y^n) = p(\alpha_n | \theta, y_n) \prod_{t=d+1}^{n-1} p(\alpha_t | \alpha_{t+1}, \theta, y^n) \prod_{t=1}^{d} p(\alpha_t | \alpha_{t+1}, \theta, y^n)\,,$$

where $d$ denotes the number of non-stationary states.[10] The first and second term are given by the Kalman smoother while the third term depends on the applied initialization of the filter, see Section 4.1.

Obtaining samples from the second term in Equation (16) is more challenging. Using the block independence assumption, we obtain that

$$\begin{aligned} p(\theta | \alpha^n, y^n) &= p(\theta | c^n, p^n, y^n) \\ &= p(\theta_p | p^n)\, p(\theta_c | c^n)\, p(\theta_{ind} | c^n, ind^n)\,. \end{aligned} \tag{17}$$

A detailed description of how to draw from each of the three conditionals in Equation (17) is given in Appendix A.5.

## 4.5 Prediction

Obtaining predictions for the filtered and smoothed state vector as well as the observation vector is straightforward.

For maximum likelihood estimation, predictions for the state vector can be made using the recursion in Equation (12) with the MLE estimates of the parameter matrices. For a forecast horizon of $h \in \mathbb{Z}$, we have that

$$a_{n+h|n} = \hat{\alpha}_{n+h} = \mathbb{E}[\alpha_{n+h} | y^n] = \mathbb{E}[T_{n+h}\alpha_{n+h-1} + R_{n+h}\varepsilon_{\alpha,n+h-1} | y^n] = T_{n+h}\alpha_{n+h-1|n},$$

[10] The number of non-stationary states $d$ depends on the model specification. For instance, if $p_t$ follows a damped trend, then d = 1.

$$P_{n+h|n} = V_{n+h} = \text{var}\left(\alpha_{n+h}|y^n\right) = T_{n+h}P_{n+h-1|n}T'_{n+h} + R_{n+h}Q_{n+h}R'_{n+h}.$$

Note that predictions for the filtered and smoothed state for points in time $n+h$ coincide. Similarly, for the observation equation, it holds that

$$\mathbb{E}\left[y_{n+h}|y^n\right] = \mathbb{E}\left[Z_{n+h}\alpha_{n+h} + \varepsilon_{y,n+h}|y^n\right] = Z_{n+h}\hat{\alpha}_{n+h},$$
$$\text{var}\left(y_{t+h}|y^n\right) = Z_{n+h}P_{n+h|n}Z'_{n+h} + H_{n+h}.$$

For Bayesian estimation, the posterior predictive densities for the state and the observation vector are given by

$$p\left(\alpha_{n+h}|y^n\right) := \int_\Theta p\left(\alpha_{n+h}|\theta, y^n\right) p\left(\theta|\alpha^n, y^n\right) d\theta,$$
$$p\left(y_{n+h}|y^n\right) := \int_{\Theta\times A} p\left(y_{n+h}|\alpha_{n+h}, \theta, y^n\right) p\left(\alpha_{n+h}|\theta, y^n\right) p\left(\theta|\alpha^n, y^n\right) d\theta d\alpha_{n+h},$$

where $\Theta$ and $A$ denote the respective parameter spaces. Intuitively, each draw from $\Theta$ produces a draw from $A$ for which prediction is straightforward using the usual recursions.

## 4.6 Example of a NAWRU specification

In this example, we consider a traditional Phillips curve with two cycle terms and additionally include one exogenous variable $z_t$, namely the wage share. We assume the cycle follows a stationary AR(2) process and impose a local linear trend.

Given Equations (2), (3) and (4), this specific NAWRU model can be rewritten in state space representation by defining the observation and state vectors and the system matrices

$$y_t = \begin{bmatrix} ur_t \\ \Delta^2 \log nulc_t \end{bmatrix} \quad Z_t = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ z_t & \beta_{c,0} & \beta_{c,1} & 0 & 0 & \mu & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\alpha_{t+1} = \begin{bmatrix} \vartheta \\ c_{t+1} \\ c_t \\ p_{t+1} \\ \eta_{t+1} \\ 1 \\ \varepsilon_{nulc,t+1} \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_{c,1} & \phi_{c,2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $Q = \text{diag}\left(\sigma_c^2, \sigma_p^2, \sigma_\eta^2, \sigma_{nulc}^2\right)$. The parameters to be estimated are

$$\theta := \left(\beta_{c,0}, \beta_{c,1}, \mu, \vartheta, \phi_{c,1}, \phi_{c,2}, \sigma_c^2, \sigma_p^2, \sigma_\eta^2, \sigma_{nulc}^2\right)$$

with restrictions

$$\beta_{c,0} < 0,$$

$$\phi_{c,2} - \phi_{c,1} < 1, \quad \phi_{c,2} + \phi_{c,1} < 1, \quad |\phi_{c,2}| < 1,$$

$$\sigma_g^2, \sigma_c^2, \sigma_p^2, \sigma_\eta^2 > 0.$$

Using **RGAP**, this model can easily be defined by

```
R> data("gap")
R> tsList <- amecoData2input(gap[["France"]], alpha = 0.65)
R> exoType <- initializeExo(varNames = "ws", D = 2, L = 0)
R> model <- NAWRUmodel(tsl = tsList, trend = "RW2", cycle = "AR2",
+                       type = "TKP", cycleLag = 0:1, exoType = exoType)
R> model
```

The function `NAWRUmodel` defines all system vectors and matrices and checks the validity of the specified model. The first input is a list of time series objects that includes the observations *ur* and *nulc* and the exogenous variable *ws*, if applicable. All remaining inputs specify the model features. For instance, `cycleLag = 0:1` means that cycle components up to lag 1 are included in the Phillips curve, i.e., $c_t$ and $c_{t-1}$. `NAWRUmodel` returns an object of class '`NAWRUmodel`' which can in turn be used to estimate the model using the function `fit`. Printing the obtained model gives an overview of the model features.

```
Call:
```

```
NAWRUmodel(tsl = tsList, trend = "RW2", cycle = "AR2", type = "TKP",
   cycleLag = 0:1, exoType = exoType)

        State space model object of class NAWRUmodel

cycle                        AR2
trend                        RW2
phillips curve
 type                        TKP
 cycle lag                   0,1
 error term                  iid normal
 exogenous variables         pcddws
anchor
 value                       -
 horizon                     -
dimensions
 number of observations      59
 period                      1962 - 2020
 frequency                   annual

Object is a valid object of class NAWRUmodel.
```

## 4.7 Example of a TFP specification

In this example, we consider a re-parameterized AR(2) cycle process and a damped trend. We assume that the CUBS error term follows an AR(1) process and that $\sigma_p^2 = 0$. Equations (7) - (10) cast into state space representations yields

$$y_t = \begin{bmatrix} \log tfp_t \\ cubs_t \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ \beta_{c,0} & 0 & 0 & 0 & \mu & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\alpha_{t+1} = \begin{bmatrix} c_{t+1} \\ c_t \\ p_{t+1} \\ \eta_{t+1} \\ 1 \\ \hat{\varepsilon}_{t+1} \end{bmatrix}, \quad T = \begin{bmatrix} 2A\cos\left(^{2\pi}/_{\tau}\right) & -A^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi_\eta & (1-\phi_\eta)\omega & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{\hat{\varepsilon}} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $Q = \text{diag}\left(\sigma_c^2, \sigma_\eta^2, \sigma_{cubs}^2\right)$. The parameters to be estimated are

$$\theta := \left(\beta_{c,0}, \mu, \omega, \phi_\eta, A, \tau, \phi_{\hat{\varepsilon}}, \sigma_c^2, \sigma_\eta^2, \sigma_{cubs}^2, \right)$$

with restrictions

$$\beta_{c,0} > 1,$$

$$|\phi_\eta|, |\phi_{\hat{\varepsilon}}| < 1,$$

$$\sigma_c^2, \sigma_\eta^2, \sigma_{cubs}^2 > 0.$$

**RGAP** offers a simple way of defining this model via

```
R> data("gap")
R> tsList <- amecoData2input(gap[["Italy"]], alpha = 0.65)
R> model <- TFPmodel(tsl = tsList, trend = "DT", cycle = "RAR2",
+                    cycleLag = 0, cubsErrorARMA = c(1, 0))
R> model
```

The function `TFPmodel` defines all state space system matrices given the specified model features. Here, `"DT"` denotes a damped trend and `"RAR2"` the re-parametrized AR(2) cycle. Moreover, the CUBS equation has an AR(1) error as indicated by `cubsErrorARMA = c(1, 0)`. A summary of the model features can be printed.

```
Call:
TFPmodel(tsl = tsList, trend = "DT", cycle = "RAR2", cycleLag = 0,
   cubsErrorARMA = c(1, 0))

        State space model object of class TFPmodel

cycle                        RAR2
trend                        DT
cubs
 cycle lags                  0
 error term                  ARMA(1,0)
 exogenous variables  -
anchor
 value                       -
 horizon                     -
dimensions
 number of observations      36
 period                      1985 - 2020
 frequency                   annual

Object is a valid object of class TFPmodel.
```

# 5 Functionality of RGAP

The EC output gap estimation with **RGAP** is done in a simple, multi-step procedure.

The first step concerns the data preparation process. **RGAP** offers several tools to compute and prepare the necessary data from a set of base data. The base data can either be provided by the user or be fetched from the AMECO database using the function `fetchAMECOdata`. The base data comprises the variables in the upper part of Table 4. Using the base data, the function `amecoData2input` computes all remaining input data for the estimation, with the exception of the CUBS indicator. Given the necessary survey and corresponding value added series, the function `cubs` computes the CUBS indicator. If the base data was retrieved using `fetchAMECOdata`, the CUBS indicator is already included. The output from `amecoData2input` and `cubs` (if not already included) amount to the input data for the EC output gap estimation. The before mentioned data related functions are intended to simplify the data preparation process, but can be neglected if the prepared data is already at hand.

In a second step, the modeling functions `NAWRUmodel` and `TFPmodel` provide a simple framework for the definition of the two unobserved component models. More precisely, they use the trend and cycle specification, and the attributes for the second observation equation to initialize the appropriate system matrices of the state space models. For the NAWRU model, the list of time series has to contain at least the unemployment rate $ur_t$ and a Phillips curve indicator variable, i.e., for the traditional Phillips curve the indicator $nulc_t$ and for the New-Keynesian type the indicator $rulc_t$. Additionally, for `NAWRUmodel`, an anchor value and horizon can be provided. Similarly, for the TFP model, the list of time series needs to contain total factor productivity $tfp_t$ and the indicator $cubs_t$. The modeling functions return objects of class '`NAWRUmodel`' and '`TFPmodel`', respectively. The model specifications can be printed via the S3 method `print`.

Third, based on the objects of class '`NAWRUmodel`' and '`TFPmodel`', the corresponding estimation function `fit` applies the Kalman filter and smoother and estimate the state space models via maximum-likelihood estimation (`method = "MLE"`) or Bayesian methods (`method = "bayesian"`). In the former case, parameter constraints can be initialized using the function `initializeRestr`. They can subsequently be modified and passed on to the optimization routines via the input variable `parRestr`. If left unspecified, the routines apply a general set of parameter constraints.[11] Using Bayesian estimation techniques, Markov chain Monte Carlo (MCMC) sampling via a nested Gibbs procedure is applied. The necessary prior distributions and complementary box constraints can be initialized using the function `initializePrior` and subsequently modified. If left unspecified, a default set of prior distributions is used.

These fitting functions are wrappers around functions in **KFAS** and provide the necessary updating routines for the involved constrained parameter estimation. The estimation functions return objects of class '`NAWRUfit`' and '`TFPfit`', respectively. **RGAP** provides S3 methods for both objects to print and plot the result. `print` displays model specifications, estimation results and basic model fit criteria. `plot` displays the smoothed state variable of interest and some residual diagnostic plots in the MLE case. Alternatively, by setting `posterior = TRUE`, posterior diagnostic plots for each parameter can be obtained. In addition, all involved observation and state series can be predicted using the S3 method `predict`.

Last, the production function output gap and potential output can be computed by passing on the estimation objects and the input data to the function `gapProd`. This function also applies the HP-filter to obtain the trend of the appropriate labor series, for which the smoothing constant can be set. **RGAP** also offers S3 printing and plotting methods for the returned object '`gap`'. For comparison, the output gap can also be computed with the standard HP filter using the function `gapHP` or an alternative unobserved component framework using `fit` (see Kuttner, 1994).

# 6 Computing the output gap with RGAP

This section details procedures to estimate the output gap using **RGAP**. The NAWRU and TFP trend are estimated using both MLE and Bayesian methods. The use of the

[11] See Section A.6 in the Appendix for details on the variance constraints. Leaving the variances unconstrained may pose an identification problem. In particular, the variability is often assigned to either the circle or the trend, leaving the respective other process with zero variance.

most important auxiliary functions is presented, for instance those concerning parameter restrictions and prior distributions.

## 6.1 Data

The data necessary to compute the output gap using the production function methodology is given in Table 4. The variables in the lower part of the table are computed based on the variables in the upper part which is done by supplying the latter to the function `amecoData2input`. The name of the function originates from the fact that all involved variables are provided by the AMECO data base. As such, the units and names of the data needs to be as specified in Table 4. The input data using the AMECO vintage 'Autumn 2018' can be retrieved by

```
R> data("gap")
R> tsList <- amecoData2input(gap[["Germany"]], alpha = 0.65)
```

The second function input `alpha` represents the output elasticity of labor. Under the EC output gap methodology, it is currently set to 0.65 for all countries.

Additionally, for the computation of the TFP trend, the CUBS indicator needs to be at hand. It can be computed with **RGAP** using the function `cubs`. Two time series lists need to be passed on to `cubs`. One with the indicator series for the industry sector, the service sector, and the building sector (see Table 5). The second one comprises the respective value added series in the same order. The indicator series can be provided at a higher frequency than the value added series and will in that case be temporally aggregated during the procedure. CUBS can be computed by

```
R> namesCubs <- c("indu","serv", "buil")
R> namesVACubs <- paste0("va", namesCubs)
R> tscubs <- cubs(tsCU = gap[["Germany"]][, namesCubs],
+                 tsVA = gap[["Germany"]][, namesVACubs])
R> tsList <- c(tsList, tscubs)
```

The package contains two data sets for illustrative purposes. The first one is called `gap` and includes the AMECO data vintage 'Autumn 2018'. All relevant variables are provided on yearly frequency, including the indicator series to compute CUBS. The second set of data is called `indicator` and contains only the indicator series at their natural frequency, i.e., either monthly or quarterly. The data is taken from the EC business and consumer surveys. For more details, see `?gap` and `?indicator`.

A fast way to retrieve the latest relevant AMECO data is via `fetchAMECOdate`. By default, the CUBS indicator is computed during the procedure. By specifying the input variable `country`, the data is only downloaded for one country. If left unspecified, a list with all available countries is returned. For instance, data for Germany can be fetched and transformed by

```
R> tslBase <- fetchAMECOdata(country = "Germany")
R> tsList <- amecoData2input(tslBase, alpha = 0.65)
```

| Variables | Short Description | Description | Unit | AMECO |
|---|---|---|---|---|
| *gdp* | real gdp | Gross domestic product at constant prices | Bn National currency | ovgd |
| *ngdp* | nominal gdp | Gross domestic product at current prices | Bn National currency | uvgd |
| *k* | capital stock | Net capital stock at constant prices: total economy | Bn National currency | oknd |
| *l* | total hours worked | Total annual hours worked: total economy | million hours | nlht |
| *et* | employment | Employment, persons: all domestic industries (National accounts) | 1000 persons | netd (fetd) |
| *etd* | domestic employment | Employment, persons: total economy (National accounts) | 1000 persons | netn |
| *eet* | employees | Employees, persons: all domestic industries (National accounts) | 1000 persons | nwtd (fwtd) |
| *popw* | population of working age | Population: 15 to 64 years | 1000 persons | npan |
| *wtotal* | total wages of employees | Compensation of employees: total economy | Bn National currency | uwcd |
| *ur* | unemployment rate | Unemployment rate: total, Member States: definition EUROSTAT | Percentage of active population | zutn |
| *pconsp* | consumption deflator | Price deflator private final consumption expenditure | National currency | pcph |
| *nulc* | nominal unit labor costs | Nominal unit labour costs: total economy (Ratio of compensation per employee to real GDP per person employed.) | National currency | plcd |
| $gdpdefl$ | gdp deflator | Price deflator gross domestic product | National currency | pvgd |
| $tfp$ | total factor productivity | Cobb-Douglas residual total factor productivity | | |
| $lfnd$ | labor force non-domestic | unit: 1000 persons | | |
| $parts$ | participation rate | labor participation rate | (ratio) | |
| $ahours$ | average hours worked | Average annual hours worked per person employed | hours | (nlha) |
| *prod* | labor productivity | labor productivity, unit: real output per person employed | M | |
| *tot* | terms of trade | Terms of trade: consumption deflator per price deflator | (ratio) | |
| *ws* | wage share | wage/labor share, unit: compensation of employees per unit of nominal output | (ratio) | |
| $winfl$ | wage inflation | growth rate of compensation of employees per employee | | |
| *rulc* | real unit labor costs | nominal unit labor costs adjusted for consumption | National currency | |

Table 4: List of variables used to compute the output gap according to the EC production function approach. The descriptions are partly taken from the AMECO data base. Domestic concept indicates that residents and non-resident who work for resident producer units are included. The top panel comprises the base data based on which the variables in the bottom panel can be computed.

| Variables | Short description | Survey |
|---|---|---|
| *indu* | Capacity utilization in industry | Quarterly industry sector survey |
| *serv* | Confidence indicator in the service industry | Monthly service sector survey |
| *buil* | Confidence indicator in the bulding and construction industry | Monthly building and construction sector survey |
| *vaindu* | Value added industry | |
| *vaserv* | Value added service industry | |
| *vabuil* | Value added construction | |

Table 5: List of variables for the computation of the capacity utilization economic sentiment indicator (CUBS).

## 6.2 Estimating the NAWRU

The NAWRU model as specified in Section 2.1 can be cast into state space representation (see Section 4.6). The first observation equation describes the unemployment rate which is decomposed into a trend (the NAWRU) and a cyclical part. The second observation equation contains the Phillips curve – either a traditional Keynesian modeling approach (TKP) or a New-Keynesian, forward looking curve (NKP).

In the example below, we compute the NAWRU for France based on the AMECO Autumn 2018 vintage. The model equations are given by

$$
\begin{aligned}
ur_t &= p_t + c_t. \\
\Delta^2 \log nulc_t &= \mu + \beta_{c,0} c_t + \vartheta \Delta^2 ws_t + \varepsilon_{nulc,t}, & \varepsilon_{nulc,t} &\sim \mathcal{N}\left(0, \sigma^2_{nulc}\right) \\
p_t &= p_{t-1} + \eta_{t-1} + \varepsilon_{p,t}, & \varepsilon_{p,t} &\sim \mathcal{N}\left(0, \sigma^2_{p}\right) \\
\eta_t &= \eta_{t-1} + \varepsilon_{\eta,t}, & \varepsilon_{\eta,t} &\sim \mathcal{N}\left(0, \sigma^2_{\eta}\right) \\
c_t &= \phi_{c,1} c_{t-1} + \phi_{c,2} c_{t-2} + \varepsilon_{c,t}, & \varepsilon_{c,t} &\sim \mathcal{N}\left(0, \sigma^2_{c}\right)
\end{aligned}
$$

i.e., the trend is modeled as second order random walk (RW2), the cycle as an AR(2) process. The model features a traditional Phillips curve that includes the second difference of the wage share. The above specification can be defined in **RGAP** by the following code:

```
R> data("gap")
R> tsList <- amecoData2input(gap[["France"]], alpha = 0.65)
R> exoType <- initializeExo(varNames = "ws", D = 2, L = 0)
R> model <- NAWRUmodel(tsl = tsList, trend = "RW2", cycle = "AR2",
+                      type = "TKP", cycleLag = 0, exoType = exoType)
```

The function `NAWRUmodel` is supplied with a time series list of data, the trend and cycle specification, and additional features of the Phillips curve such as its type, exogenous variables and the number of cycle lags that should be included. The function `initializeExo` returns a three dimensional array that allows the user to specify the difference and lag order of the exogenous variables.

```
R> exoType
, , difference

     ws
[1,]  2

, , lag

     ws
[1,]  0
```

The columns of the array represent the exogenous variables that are included in the model. The number of rows represent different transformations. `NA` entries are simply ignored. Here, only one variable and only one specification of that variable is included: the second difference of the wage share.

In order to estimate the model using MLE, some parameter restrictions are necessary. Stationarity restrictions are applied to the appropriate variables if no box constraints are specified. The function `initializeRestr` initializes the box constraints and returns a list of matrices with constraints for each equation, i.e., the cycle, the trend and the second observation equation.[12]

```
R> parRestr <- initializeRestr(model = model, type = "hp")
R> parRestr
$cycle
   cPhi1 cPhi2     cSigma
LB    NA    NA 0.01276674
UB    NA    NA 0.51927755

$trend
   tSigma     tdSigma
LB      0 0.001861214
UB      0 0.081948136
```

[12]The names of the parameters are chosen such that they are easily assignable to the respective equations. For instance, `cPhi1` and `cPhi2` correspond to the autoregressive parameters of the cycle equation $\phi_{c,0}, \phi_{c,1}$. `tSigma` denotes the trend variance $\sigma_p^2$ and `tdSigma` the variance of the trend drift $\sigma_\eta^2$. For the Phillips curve equation `pcInd` (or second observation equation `E2`), `pcConst` denotes the constant $\mu$ and `pcC0` the parameter on the contemporaneous cycle $\beta_{c,0}$. The coefficient of the exogenous variable $\Delta^2 ws_t$ is denoted by `pcddws`.

```
$pcInd
   pcddws pcConst pcC0      E2Sigma
LB     NA      NA   NA 1.950937e-05
UB     NA      NA   NA 8.716629e-04
```

NA indicates that no additional box constraints are applied, though the stationarity constraints remain in place (if applicable). The list can be modified before supplying it to the fitting function `fit` alongside the model object. `fit` performs filtering and smoothing, and estimates the parameters of the supplied state space model by MLE.

```
R> f <- fit(model = model, parRestr = parRestr)
R> plot(f)
```

After the optimization is done, the model specification as well as the estimation results are printed.

```
Call:
fit.NAWRUmodel(model = model, parRestr = parRestr)

        State space model object of class NAWRUmodel

cycle                       AR2
trend                       RW2
phillips curve
 type                       TKP
 cycle lag                  0
 error term                 iid normal
 exogenous variables        pcddws
anchor
 value                      -
 horizon                    -
dimensions
 number of observations     59
 period                     1962 - 2020
 frequency                  annual


        Maximum likelihood estimation results

cycle
          Coefficient Standard Error t-statistic  p-value
 cPhi1        1.254         0.1291       9.707 0.00e+00
 cPhi2       -0.393         0.4645      -0.846 3.98e-01
 cSigma       0.208         0.0404       5.164 2.42e-07

trend
          Coefficient Standard Error t-statistic p-value
 tdSigma     0.00227         0.0017        1.33   0.182

phillips curve
          Coefficient Standard Error t-statistic  p-value
 pcC0      -0.003575       1.52e-03     -2.3510 1.87e-02
 pcConst    0.000045       2.05e-03      0.0219 9.83e-01
 pcddws     0.985955       9.70e-02     10.1672 0.00e+00
```

```
 pcSigma    0.000122       2.28e-05      5.3417 9.21e-08
 RMSE: 0.0116
 R2: 0.628
 Box-Ljung test: X-squared = 20.9, df = 10, p-value = 0.022

loglik            AIC             BIC             HQC
138.8665       -261.7330       -245.1127       -255.2452
signal-to-noise
0.0109
```

`fit` returns an object of class '`NAWRUfit`' for which S3 methods provide some plots, see Figures 1 and 2. The diagnostic plots of the Phillips curve suggest that the model fit is quite satisfactory. There is no autocorrelation in the recursive residuals and the histogram suggests that they are approximately normal. On a 10% level, the $H_0$ of the Box-Ljung test cannot be rejected, indicating that the residuals are independent. The $R^2$ is equal to 0.63, underpinning the good model fit. Figure 1 shows the unemployment rate and the estimated NAWRU alongside 95% confidence intervals. From the beginning of the sample in 1962 until the turn of the millennium, the NAWRU for France has been steadily increasing. Thereafter, it has remained on a relatively high plateau of a little over 9%.

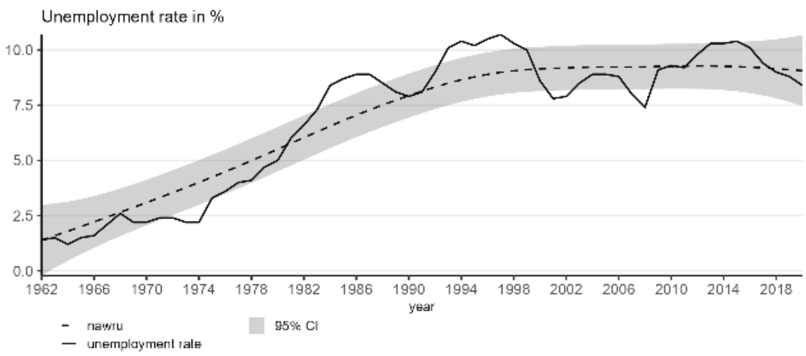


Figure 1: The unemployment rate and the estimated NAWRU for France.
*Notes:* The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The NAWRU follows a local linear trend and the corresponding cycle an AR(2) process. The traditional Phillips curve includes a contemporaneous cycle term and the second difference of the wage share. The shaded area denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

The object '`NAWRUfit`' is a list that contains several items regarding model fit, the resulting time series, the used parameter restrictions, the estimated parameters, and the '`NAWRUmodel`' object itself, among others.

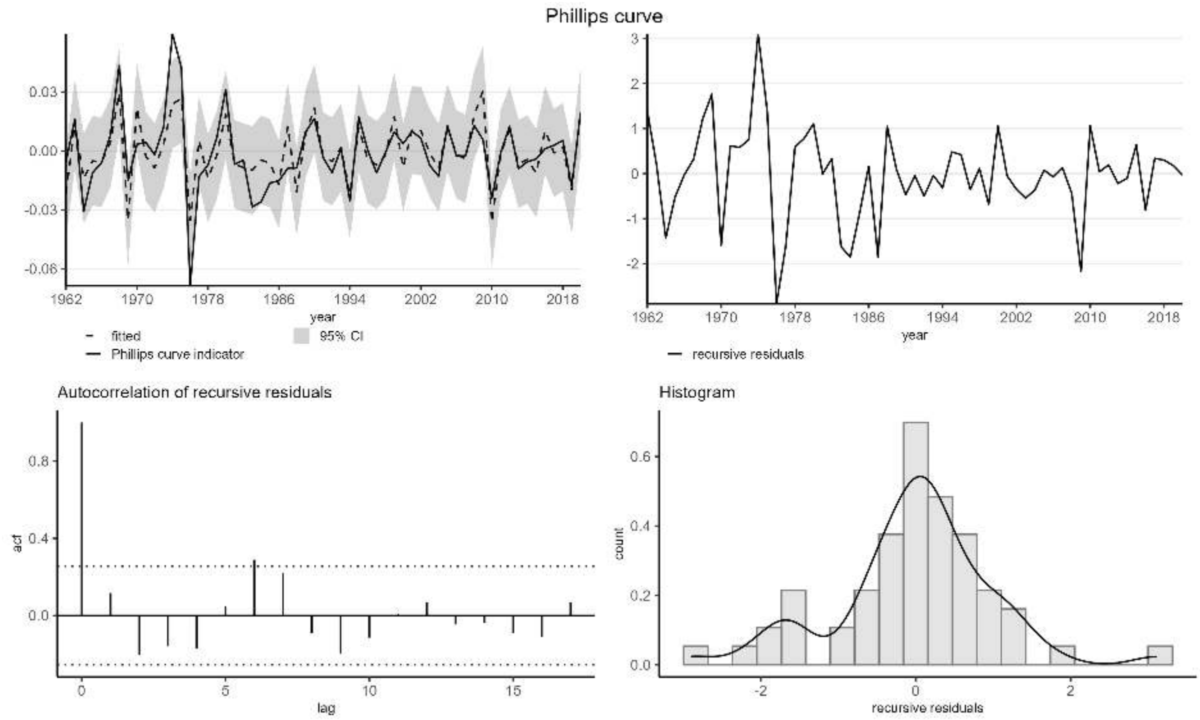


Figure 2: The Phillips curve indicator and diagnostic plots for France.
*Notes:* The top left panel shows the Phillips curve, the top right panel the recursive residuals, the bottom left panel their autocorrelation and the bottom right panel the corresponding histogram. The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The NAWRU follows a local linear trend and the corresponding cycle an AR(2) process. The traditional Phillips curve includes a contemporaneous cycle term and the second difference of the wage share. The shaded area in the top left panel denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

When defining the model using `NAWRUmodel`, an anchor and its corresponding horizon can be supplied. Alternatively, the anchored series can be computed based on the previously fitted object `f` by

```
R> f <- trendAnchor(fit = f, anchor = 8, h = 10, returnFit = TRUE)
R> plot(f)
```

The anchored NAWRU is shown in Figure 3.

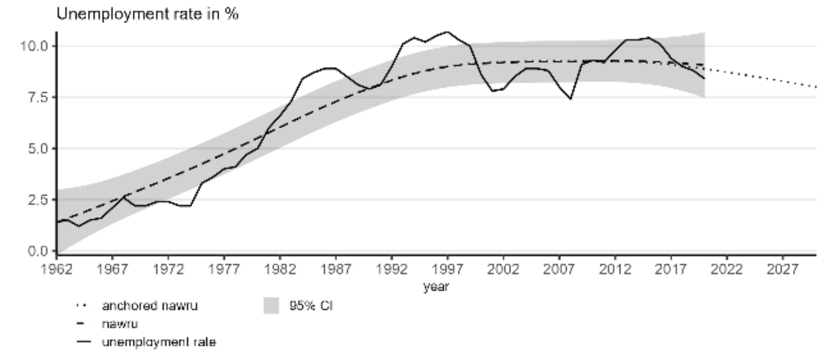


Figure 3: The unemployment rate, the estimated NAWRU, and the anchored NAWRU for France.
*Notes:* The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The NAWRU follows a local linear trend and the corresponding cycle an AR(2) process. The traditional Phillips curve includes a contemporaneous cycle term and the second difference of the wage share. The anchor is set to 8% after 10 years. The shaded area denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

## 6.3 Estimating the TFP trend

For the estimation of the trend of TFP, a second state space model is fitted. To that end, the TFP model from Section 2.2 is cast into state space representation and estimated by the use of the Kalman filter. As for the NAWRU, the TFP trend splits into a permanent component (its trend) and a cyclical one. The CUBS indicator helps to separate trend and cycle via a second observation equation.

We compute the TFP trend for Italy based on the AMECO Autumn 2018 vintage. The model equations are given by

$$\begin{aligned}
\log tfp_t &= p_t + c_t, \\
cubs_t &= \mu + \beta_{c,0} c_t + \varepsilon_{cubs,t}, & \varepsilon_{cubs,t} &\sim \mathcal{N}\left(0, \sigma^2_{cubs}\right) \\
p_t &= p_{t-1} + \eta_{t-1}, \\
\eta_t &= (1-\phi_\eta)\omega + \phi_\eta \eta_{t-1} + \varepsilon_{\eta,t}, & \varepsilon_{\eta,t} &\sim \mathcal{N}\left(0, \sigma^2_\eta\right) \\
c_t &= 2A\cos\left(2\pi/\tau\right) c_{t-1} - A^2 c_{t-2} + \varepsilon_{c,t}, & \varepsilon_{c,t} &\sim \mathcal{N}\left(0, \sigma^2_c\right)
\end{aligned}$$

i.e., the CUBS equation only includes the contemporaneous cycle, the trend is modeled as a damped trend, and the cycle as a re-parametrized AR(2) process. This specification can be modeled and fitted in **RGAP** by

```
R> data("gap")
R> tsList <- amecoData2input(gap[["Italy"]], alpha = 0.65)
```

```
R> model <- TFPmodel(tsl = tsList, trend = "RW2", cycle = "RAR2",
+                     cycleLag = 0, cubsErrorARMA = c(0,0))
R> parRestr <- initializeRestr(model = model, type = "hp")
R> f <- fit(model = model, parRestr = parRestr)
R> plot(f)
```

After the optimization is completed, the model setup and the results are printed.

```
Call:
fit.TFPmodel(model = model, parRestr = parRestr)

        State space model object of class TFPmodel

cycle                       RAR2
trend                       DT
cubs
 cycle lags                 0
 error term                 iid normal
 exogenous variables        -
anchor
 value                      -
 horizon                    -
dimensions
 number of observations     36
 period                     1985 - 2020
 frequency                  annual


Maximum likelihood estimation results

cycle
        Coefficient Standard Error t-statistic  p-value
 cA        5.35e-01       1.53e-01        3.49 0.000479
 cSigma    8.75e-05       2.27e-05        3.86 0.000115
 cTau      8.78e+00       3.39e+00        2.59 0.009516

trend
        Coefficient Standard Error t-statistic  p-value
 tdOmega    5.14e-03       4.19e-03        1.23   0.219
 tdPhi      9.16e-01       8.50e-02       10.78   0.000
 tdSigma    6.32e-06       4.93e-06        1.28   0.200

cubs
        Coefficient Standard Error t-statistic  p-value
 cuC0       2.697632       0.281942       9.568 0.00000
 cuConst    0.002346       0.009228       0.254 0.79932
 cuSigma    0.000157       0.000055       2.860 0.00423
 RMSE: 0.029
 R2: 0.363
 Box-Ljung test: X-squared = 2.15, df = 7.2, p-value = 0.957

    loglik           AIC           BIC           HQC
 192.83313    -367.66627    -353.41460    -362.69205
signal-to-noise
0.07224
```

The $H_0$ of the Box-Ljung test cannot be rejected, indicating that the residuals are independent. With 36 observations, the sample is rather small. Yet, the $R^2$ of 0.36 for the CUBS equation is satisfactory. The diagnostic plots of the CUBS equation in Figure 5 suggest that there is no autocorrelation in the recursive residuals, further supporting the model choice. Figure 4 shows the growth rate of TFP and its estimated trend with 95% confidence intervals. While the TFP trend growth has been positive until 2002, it turns negative thereafter and only recovers into positive territory in 2016, yet with increased uncertainty, as indicated by the wide confidence intervals at the end of the sample.

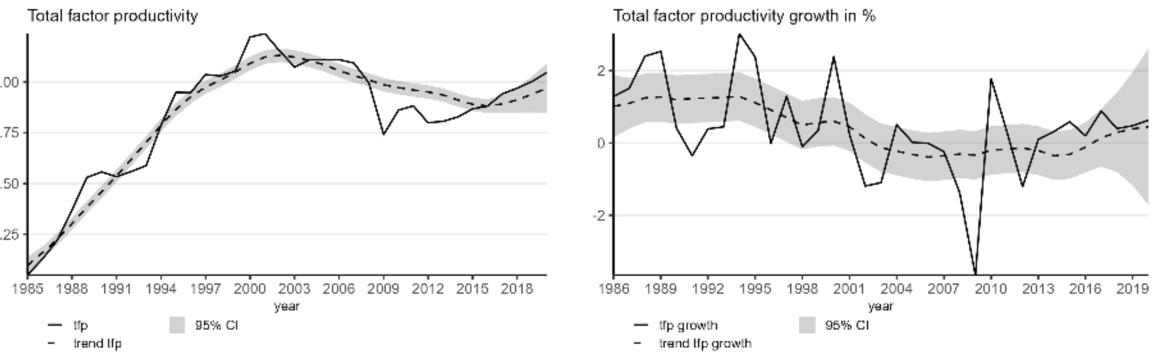


Figure 4: Total factor productiocity and its estimated trend for Italy.
*Notes:* The left panel shows the level series and the right panel the growth rate in percent. The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The permanent component of the TFP follows a damped trend and the corresponding cycle a re-parameterized AR(2) process. The CUBS equation features a contemporaneous cycle term. The shaded area denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

Predictions can be made using the function `predict` for which specific S3 plotting methods are available (see Figure 6).

```
R> fPred <- predict(object = f, n.ahead = 10)
R> plot(fPred, alpha = 0.1)
```

Alternatively, Bayesian methods can be used to estimate the TFP model. To that end, prior distributions and additional parameter restrictions need to be specified via the function `initializePrior`. They can subsequently be modified manually. All prior distributions are defined through their mean and standard deviation. All variance priors are Inverse-Gamma distributions. If the cycle is reparametrized, its parameters $A$ and $\tau$ follow the Beta distribution. All remaining priors are normal.

```
R> prior <- initializePrior(model = model)
R> prior
```

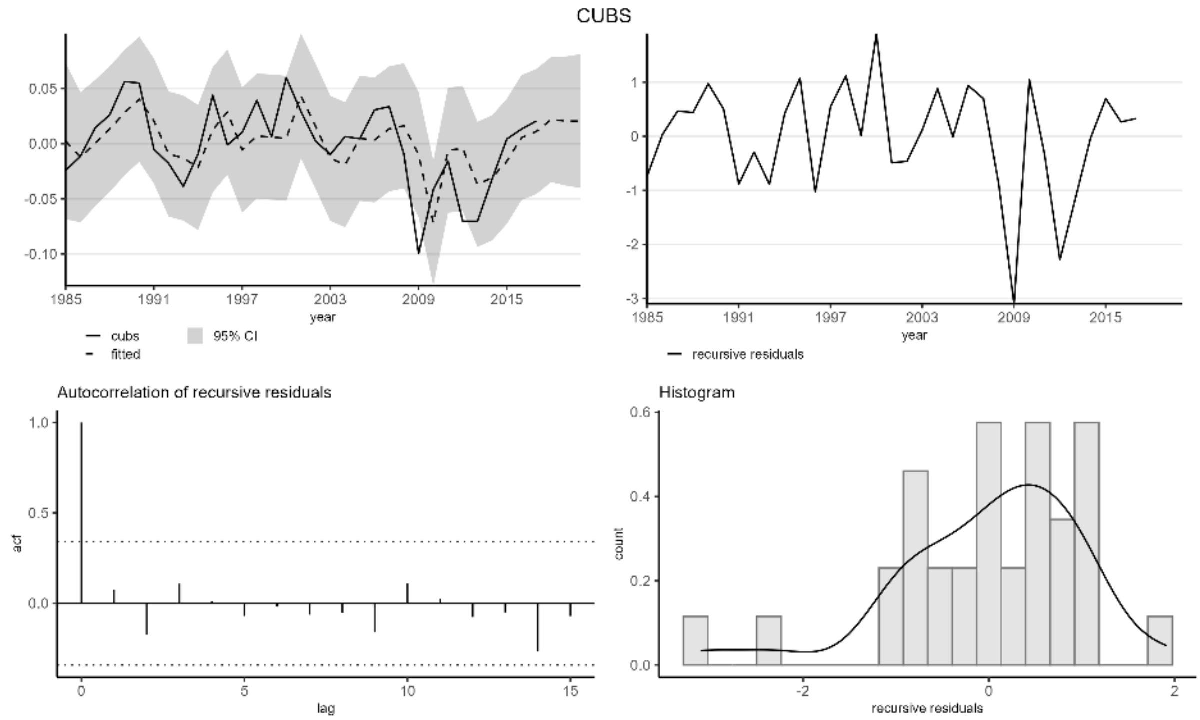


Figure 5: The CUBS equation and diagnostic plots for Italy.
*Notes:* The top left panel shows the CUBS and its fitted values, the top right panel the recursive residuals, the bottom left panel their autocorrelation and the bottom right panel the corresponding histogram. The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The permanent component of the TFP follows a damped trend and the corresponding cycle a re-parameterized AR(2) process. The CUBS equation features a contemporaneous cycle term. The shaded area in the top left panel denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

```
$cycle
       cA  cTau       cSigma
mean 0.42  8.00 3.825118e-05
std  0.17  3.50 3.825118e-05
LB   0.01  2.01 0.000000e+00
UB   0.99 31.99           NA

$trend
     tdOmega tdPhi     tdSigma
mean   0.015  0.80 1.46797e-06
std    0.010  0.24 1.46797e-06
LB        NA  0.00 0.00000e+00
UB        NA  0.99          NA

$cubs
     cuConst cuC0      E2Sigma
mean    0.00  1.4 0.0007073872
std     0.03  0.7 0.0007073872
LB        NA   NA 0.0000000000
UB        NA   NA           NA
```

The above initialization is based on the suggestions by the EC (Havik et al., 2014). Only for the variances, we deviate and use a similar procedure as for MLE. Alternatively,

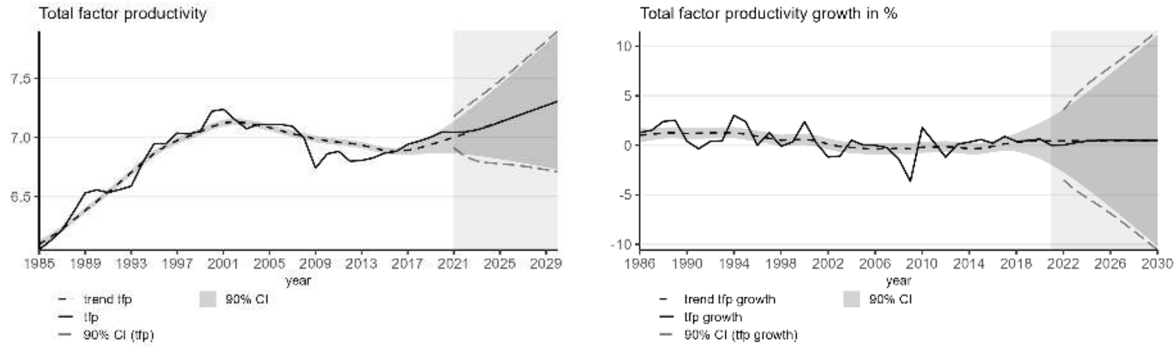


Figure 6: Prediction of total factor productivity and its estimated trend for Italy.
*Notes:* The left panel shows the level series and the right panel the growth rate in percent. The shaded area denotes the forecast window. The underlying unobserved component model was estimated using the Kalman filter and maximum likelihood estimation. The permanent component of the TFP follows a damped trend and the corresponding cycle a re-parameterized AR(2) process. The CUBS equation features a contemporaneous cycle term. The shaded area denotes the 95% confidence interval. The AMECO Autumn 2018 vintage was used.

by setting `MLE = TRUE` and additionally supplying the MLE fitted object `fit`, the prior distributions can be initialized using the previously estimated MLE coefficients.

Having specified the prior distributions, an MCMC procedure is performed via the function `fit` by setting `method = "bayesian"`.

```
R> fBayes <- fit(model = model, method = "bayesian", prior = prior,
+                R = 5000, thin = 2, MLEfit = f)
R> plot(f, posterior = TRUE)
```

In our example, the fitted MLE object `fit` is supplied to the fitting function for initialization. After the MCMC procedure is done, the model specification alongside the parameter estimates (e.g. posterior means) are printed. For each parameter, its estimate (e.g. mean), standard deviation and lower and upper bound of the chosen highest posterior density interval (HPDI) are printed. Similar to the MLE fitted object, the returned object of class '`TFPfit`' is a list containing various settings and time series as well as the applied prior distributions and Geweke convergence test results (Geweke, 1991). The procedure `fit` returns

```
Call:
fit.TFPmodel(model = model, method = "bayesian", prior = prior, R = 5000,
   thin = 2, MLEfit = fit)

        State space model object of class TFPmodel

cycle                        RAR2
trend                        DT
```

```
cubs
 cycle lags                  0
 error term                  iid normal
 exogenous variables         -
anchor
 value                       -
 horizon                     -
dimensions
 number of observations      36
 period                      1985 - 2020
 frequency                   annual


        MCMC estimation results

cycle
           Mean   Median 85% HPDI-LB 85% HPDI-UB
 cA     0.416167 0.406912    1.68e-01    6.57e-01
 cSigma 0.000158 0.000126    6.29e-05    2.23e-04
 cTau   7.997208 7.442340    3.29e+00    1.27e+01

trend
            Mean   Median 85% HPDI-LB 85% HPDI-UB
 tdOmega 6.61e-03 6.44e-03    1.08e-03    1.16e-02
 tdPhi   9.81e-01 9.84e-01    9.73e-01    9.90e-01
 tdSigma 6.52e-07 6.13e-07    3.62e-07    8.91e-07

cubs
            Mean   Median 85% HPDI-LB 85% HPDI-UB
 cuC0    1.40e+00 1.40e+00    1.382999    1.425066
 cuConst 1.62e-05 5.38e-06   -0.000842    0.000920
 cuSigma 4.36e-04 4.18e-04    0.000274    0.000558
         MRMSE signal-to-noise
      0.033540        0.004114
```

The option `posterior = TRUE` when plotting returns diagnostic plots for each parameter. Figure 7 exemplifies such diagnostic plots for the cycle variance $\sigma_c^2$. The trace plot (lower right) suggests that for this parameter, the chain has converged, while the plot of the ACF (lower right) indicates that the draws are not auto-correlated and thus that the applied thinning was sufficient.

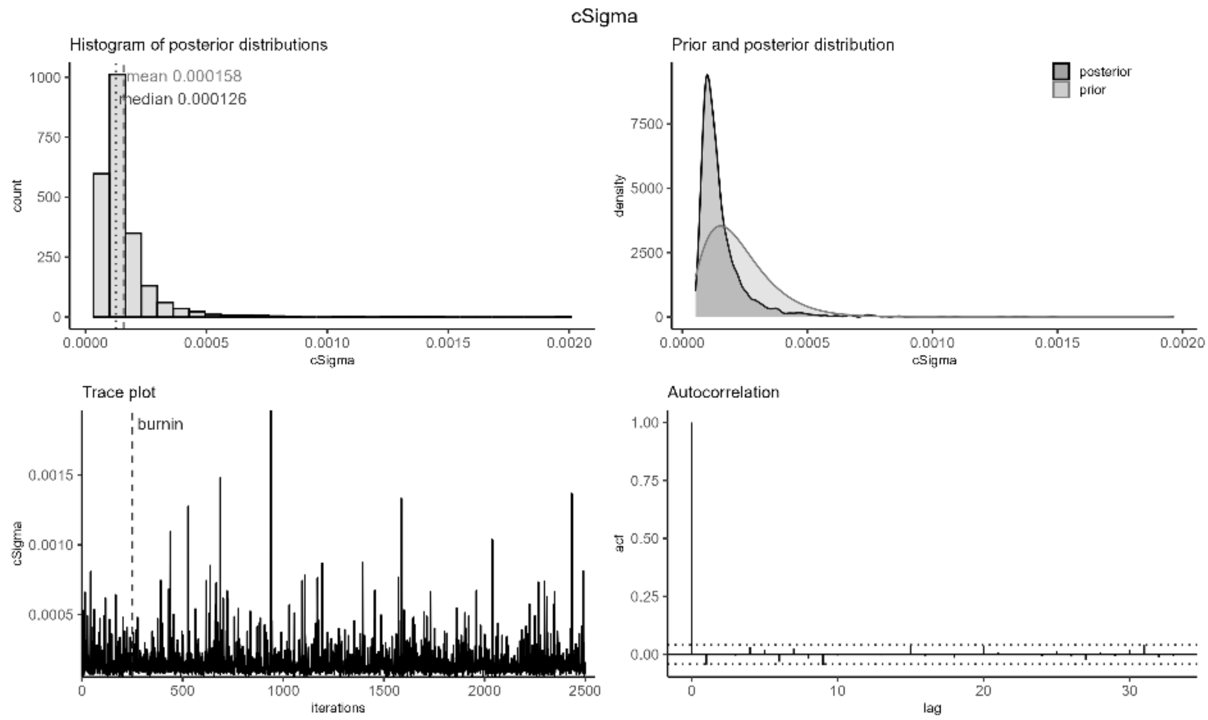


Figure 7: Posterior diagnostic plots for the cycle variance $\sigma_c^2$ in the TFP trend estimation for Italy using Bayesian methods.
*Notes:* The top left panel shows a histogram of the posterior draws, the top right panel the prior and posterior densities, the bottom left panel a trace plot of the draws and the bottom right panel the autocorrelation of the draws. The underlying unobserved component model was estimated using the Kalman filter and a Bayesian MCMC procedure. The cycle process follows a re-parameterized AR(2) process.

## 6.4 Estimating the output gap

This example deals with the output gap estimation for the Netherlands. For the NAWRU model we choose an AR(2) cycle, a local linear trend, and a TKP curve with six exogenous variables. The TFP model contains a damped trend, a re-parameterized AR(2) cycle, and a standard CUBS equation. The following code specifies the two models in **RGAP**:

```
R> data("gap")
R> tsList <- amecoData2input(gap[["Netherlands"]], alpha = 0.65)
R> model <- parRestr <- prior <- fit <- list()
R> D <- matrix(c(2, 2, 2, 1, 1, 1), 2, 3, byrow = TRUE)
R> L <- matrix(c(0, 0, 0, 1, 1, 1), 2, 3, byrow = TRUE)
R> exoType <- initializeExo(varNames = c("ws", "prod","tot"), D = D, L = L)
R> model$nawru <- NAWRUmodel(tsl = tsList, trend = "RW2", cycle = "AR2",
+                            type = "TKP", cycleLag = 0, exoType = exoType)
R> model$tfp <- TFPmodel(tsl = tsList, trend = "DT", cycle = "RAR2",
+                        cycleLag = 0, cubsErrorARMA = c(0,0))
```

We estimate both models using MLE and compute the output gap using the function `gapProd`:

```
R> parRestr$nawru <- initializeRestr(model = model$nawru, type = "hp")
R> fits$nawru <- fit(model = model$nawru, parRestr = parRestr$nawru)
R> parRestr$tfp <- initializeRestr(model = model$tfp, type = "hp")
R> fits$tfp <- fit(model = model$tfp, parRestr = parRestr$tfp)
R> fits$gap <- gapProd(tsl = tsList, NAWRUfit = fits$nawru,
R>                     TFPfit = fits$tfp, lambda = 100, alpha = 0.65)
R> plot(fits$gap)
```

`gapProd` returns an object of class '`gapProd`' that consists of the fitted NAWRU and TFP objects as well as a time series list which contains all original and estimated time series. Moreover, an S3 plotting method is available for '`gapProd`'. Figure 8 presents potential GDP growth (left panel) and the output gap (right panel) for the Netherlands. Potential output was well above 2% prior to the dot-com bubble and declined considerably thereafter. After the financial crisis hit in 2008, another decline can be observed and only recently, the gap returned to levels of around 1%. The corresponding output gap appears very cyclical and over the past decades, negative gaps have become more extreme.

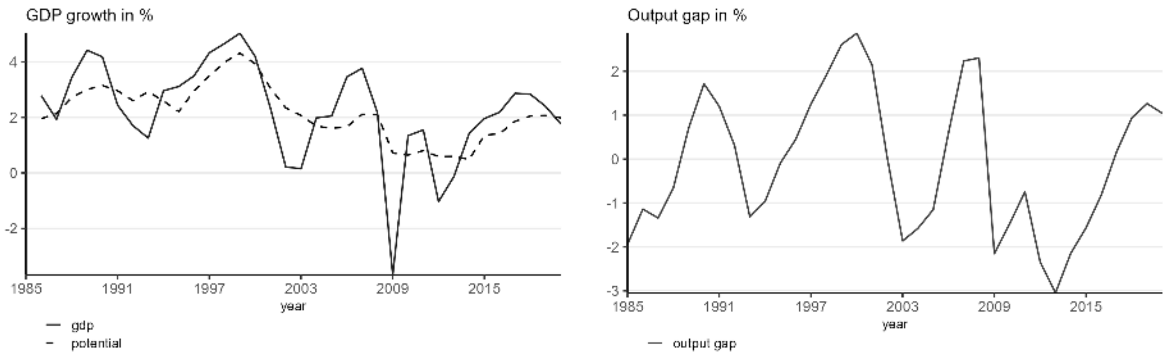


Figure 8: GDP and potential output growth, and the corresponding output gap for the Netherlands.
*Notes:* The two underlying unobserved component models were estimated using the Kalman filter and maximum likelihood estimation. The AMECO Autumn 2018 vintage was used.

Optionally, the contributions to both potential growth and the output gap can be plotted by

```
R> plot(fits$gap, contribution = TRUE)
```

The output is shown in Figure 9. The contribution of average hours worked to potential growth was negative until 2011 and positive thereafter. The working population, total factor productivity and the capital stock show a sizable and positive contribution. For the output gap, the contribution of the capital stock is zero since the model imposes that it coincides with its trend. The negative gap preceding the Global Financial Crisis of 2007-2008 is mainly accounted for by total factor productivity, which is then superseded by the negative contribution of the working population.

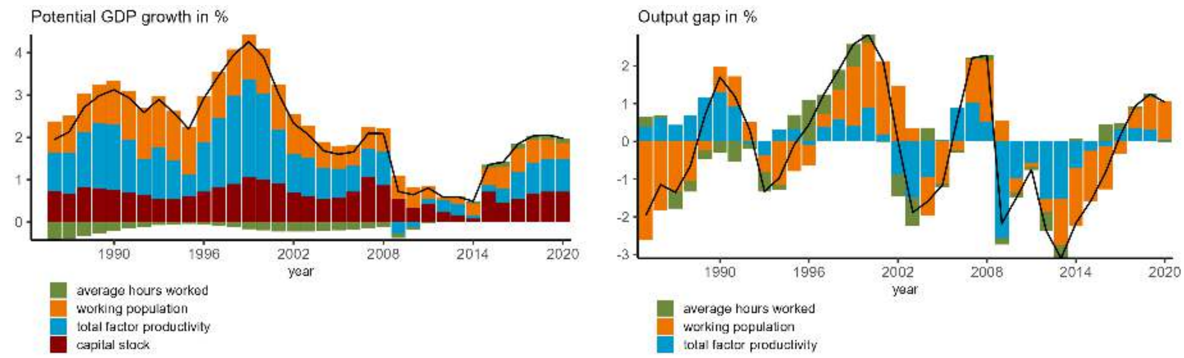


Figure 9: Contributions to potential GDP and to the corresponding output gap for the Netherlands.
*Notes:* The two underlying unobserved component models were estimated using the Kalman filter and maximum likelihood estimation. The AMECO Autumn 2018 vintage was used.

Additionally, RGAP provides two alternative estimation procedures detailed in Section 3. Kuttner's model can be specified and estimated by

```
R> tsList <- as.list(gap[["Netherlands"]][,c("cpih","gdp")])
R> tsList$infl <- diff(tsList$cpih)
R> model <- KuttnerModel(tsl = tsList, trend = "RW2",
+                         cycleLag = 1, cycle = "AR2")
R> parRestr <- initializeRestr(model = model, type = "hp", q = 0.1)
R> gapKuttner <- fit(model = model, parRestr = parRestr)
```

The input parameter `q = 0.1` indicates the quantile of the Inverse-Gamma distribution used to specify bounds for the variance parameters. `KuttnerModel` and `fit` return objects of class '`KuttnerModel`' and '`KuttnerFit`', respectively, for which S3 printing methods are available. As for previous models, plots regarding model fit and the resulting time series are also accessible via S3 plotting methods.

The HP-filtered output gap can be obtained using `gapHP`, for which the smoothing parameter `lambda` can optionally be set:

```
R> gapHPfilter <- gapHP(tsList$gdp, lambda = 100)
```

The function `gapHP` returns an object of class '`gapHP`' for which potential output growth and the output gap can be plotted using S3 methods.

Figure 10 shows the output gaps computed with the presented methods, i.e., EC the production function methodology (dashed), Kuttner's model (dotted), and the HP-filter (solid). Although the pathway of the three output gaps is similar, the differences are at times sizable, underlining the influence of model choice and parameters.

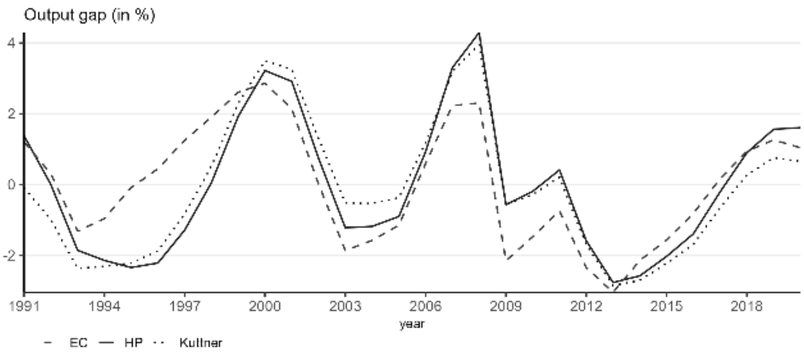


Figure 10: Comparison of different output gap estimation methods for the Netherlands.
*Notes:* The underlying unobserved component models were estimated using the Kalman filter and maximum likelihood estimation. The smoothing constant for the HP-filtered gap is set to $\lambda = 10$. The AMECO Autumn 2018 vintage was used.

Last, **RGAP** enables the fast computation of the output gap for a set of different countries. We consider the following set of countries:

```
R> countries <- c("Belgium", "Denmark", "Finland", "France",
+                 "Germany", "Greece", "Italy", "Luxembourg",
+                 "Netherlands", "Portugal", "Sweden", "United Kingdom")
```

For each country `k` in `countries`, we run

```
R> tsl <- amecoData2input(gap[[k]])
R> fit <- autoGapProd(tsl = tsl, type = "hp", fast = TRUE, nModels = 5, q = 0.1)
```

The function `autoGapProd` choses the best NAWRU and TFP trend model according to the Bayesian Information Criterion (BIC). The parameter `fast = TRUE` indicates that there is a fast pre-selection of `nModels` suitable models by first estimating the trend and cycle series by the HP-filter and subsequently choosing the best models equation by equation. The resulting subset of models is then estimated in the usual state space framework. For the initialization of the parameter restrictions, `type` is passed on to the function `initializeRestr`. Finally, the best models according to the BIC are chosen and the output gap is computed.[13] If `fast = FALSE`, a variety of possible models is estimated

[13]The set of tested models is extensive but not exhaustive. The best model is solely based on convergence and the chosen criterion (BIC or RMSE). A manual check of the results is therefore highly recommended.

via the Kalman filter, which greatly increases computing time.

Figure 11 shows the resulting output gaps. Though there are substantial differences between countries, to some extend, the gaps move in tandem. Most strikingly, all but two output gaps turn negative in 2009 after the global Financial crisis of 2008. For Greece, the crisis triggered a decade-long recession with large negative gaps. The gaps for some of the remaining countries return to positive territory only for the second half of the 2020 decade.

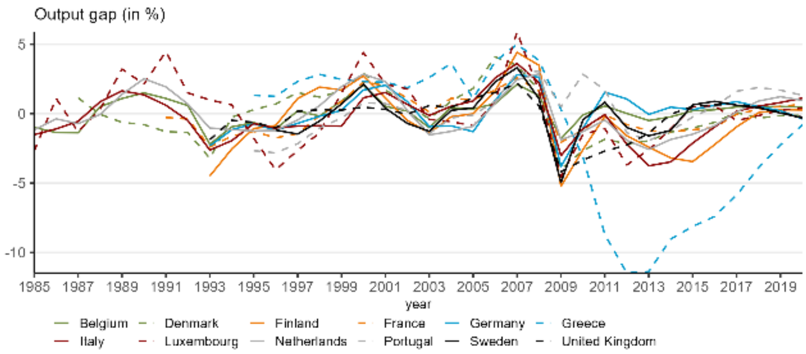


Figure 11: Comparison of the production function output gap for different countries.
*Notes:* The underlying unobserved component models were estimated using the Kalman filter and maximum likelihood estimation. The AMECO Autumn 2018 vintage was used.

# 7 Comparison to GAP Version 5.0 of the EC

We will now briefly compare **RGAP** and the software GAP Version 5.0 provided by the EC. In GAP, the data, model specification and parameter restrictions are set using an Excel interface. All estimations are done in Fortran and for graphical output related to Bayesian analysis a Matlab-based interface is used.

We re-estimate the NARWU specification detailed in Section 6.2 with the same parameter restrictions as in GAP. Figure 12 shows the NAWRU estimated with **RGAP** (solid) and GAP (dashed), respectively. The parameter estimates are shown in Table 6. The differences between the two trend estimates and the corresponding parameter estimates are negligible.[14]

[14]One possible cause for differences in the results are differences in the applied transformation functions which deploy parameter constraints. For instance, for the constraints on the autoregressive cycle

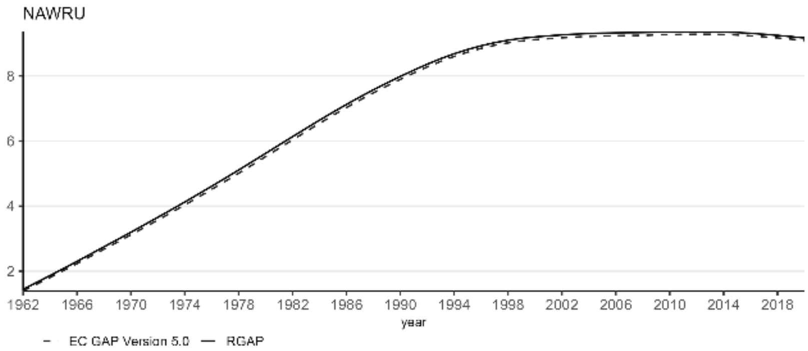


Figure 12: Comparison between the NAWRU using RGAP and EC GAP Version 5.0.
*Notes:* The NAWRU model specifications and parameter restrictions are as in Section 6.2. The AMECO Autumn 2018 vintage was used.

| | | **RGAP** | GAP Version 5.0 |
|---|---|---|---|
| cPhi1 | $\phi_{c,0}$ | 1.2637 | 1.2618 |
| cPhi2 | $\phi_{c,1}$ | −0.4050 | −0.3980 |
| cSigma | $\sigma^2_c$ | 2.097e−01 | 2.098e−01 |
| E2Sigma | $\sigma^2_{nulc}$ | 1.216e−04 | 1.217e−04 |
| pcC0 | $\beta_{c,0}$ | −0.0035 | −0.0035 |
| pcConst | $\mu$ | −0.0011 | 5.024e−05 |
| pcddws | $\vartheta$ | 1.0087 | 0.9859 |
| tdSigma | $\sigma^2_\eta$ | 1.861e−03 | 2.008e−03 |

Table 6: Comparison between the NAWRU parameter estimates using RGAP and EC GAP Version 5.0.
*Notes:* The NAWRU model specifications and parameter restrictions are as in Section 6.2. The AMECO Autumn 2018 vintage was used.

# 8 Concluding remarks

Output gap estimation is essential in the identification of the current economic position in the business cycle. In this paper, we introduce the `R` package **RGAP** with focus on the Cobb-Douglas production function methodology as suggested by the EC. The focus of this package is to provide a reliable, transparent, and feasible setting for the estimation of the output gap and potential output. The package functionalities cover data processing, model definition and estimation, prediction, as well as tailored plotting options for the results. The modeling framework is kept general such that a variety of model specifications are captured. The trend of total factor productivity and the non-accelerating wage-rate of unemployment can be retrieved via two unobserved component models that are cast in state space representation and estimated using the Kalman filter, either by maximum likelihood or Bayesian methods. All other trend series are retrieved

parameters, **GAP** imposes box constraints while **RGAP** imposes stationarity constraints.

through the HP-filter. The resulting estimates of potential output and the output gap can be decomposed into the contributions of input factors regarding labor, capital and productivity, which facilitates an economic interpretation and thus policy-making.

The most prominent advantage of the package **RGAP** – compared to the existing software by the European Commission – is its simple and flexible handling in the `R` environment for both beginners and advanced `R` users. Pre- and postprocessing of involved data can be easily incorporated. The data supply and preparation processes embedded in **RGAP** enable its usage without any other input for a wide range of countries.

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# A Appendix

## A.1 Phillips curve

Without loss of generality, we consider a hybrid form of the Phillips curve, i.e.,

$$\pi_t = \lambda^b \pi_{t-1} + \lambda^f \mathbb{E}_t \left[\pi_{t+1}\right] + \theta c_t + \varepsilon_{\pi,t}, \tag{18}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_{c,t}, \tag{19}$$

where for notational simplicity, $\pi_t$ is the Phillips curve indicator and

$$\lambda^b = \frac{\xi}{1+\delta\xi}, \quad \lambda^f = \frac{\delta}{1+\delta\xi},$$

where $\xi \in (0,1)$ denotes the degree of backwardness and $\delta \in (0,1)$ the discount factor on future consumption. Rewriting Equation (19) yields

$$\begin{pmatrix} c_t \\ c_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}}_{:=\Phi} \begin{pmatrix} c_{t-1} \\ c_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{c,t} \\ 0 \end{pmatrix}$$

and we know that for the eigenvalues $\lambda$ of $\Phi$ it holds that $|\lambda| < 1$ since $c_t$ is stationary. Hence, $I_2 - \Phi L$ is invertible. Moreover, from Equation (18) we get that

$$\left(1 - \lambda^b L - \lambda^f L^{-1}\right) \pi_t = \begin{pmatrix} \theta & 0 \end{pmatrix} \begin{pmatrix} c_t \\ c_{t-1} \end{pmatrix} + \varepsilon_{\pi,t}.$$

It can be readily shown that

$$\left(1 - \lambda^b L - \lambda^f L^{-1}\right) = \frac{\lambda^f}{\delta} \left(1 - \xi L\right) \left(1 - \delta L^{-1}\right)$$

and thus the roots of the polynomial $\left(1 - \lambda^b z - \lambda^f z^{-1}\right)$ are given by

$$z_1 = \delta < 1, \quad z_2 = \frac{1}{\xi} > 1. \tag{20}$$

Using the fact that $(1 - \delta L^{-1})$ is invertible due to Equation (20), we obtain

$$\left(1 - \xi L\right) \left(1 - \delta L^{-1}\right) \pi_t = \frac{\delta}{\lambda^f} \begin{pmatrix} \theta & 0 \end{pmatrix} \begin{pmatrix} c_t \\ c_{t-1} \end{pmatrix} + \frac{\delta}{\lambda^f} \varepsilon_{\pi,t}$$

$$\Leftrightarrow \qquad (1-\xi L)\,\pi_t = \frac{\delta}{\lambda^f}\begin{pmatrix}\theta & 0\end{pmatrix}\sum_{i=0}^{\infty}\delta^i L^{-i}\begin{pmatrix}c_t \\ c_{t-1}\end{pmatrix} + \frac{\delta}{\lambda^f}\sum_{i=0}^{\infty}\delta^i L^{-i}\varepsilon_{\pi,t}$$

$$= \frac{\delta}{\lambda^f}\begin{pmatrix}\theta & 0\end{pmatrix}\sum_{i=0}^{\infty}\delta^i \mathbb{E}_t\begin{bmatrix}c_{t+i} \\ c_{t+i-1}\end{bmatrix} + \frac{\delta}{\lambda^f}\sum_{i=0}^{\infty}\delta^i \mathbb{E}_t\left[\varepsilon_{\pi,t+i}\right]$$

$$= \frac{\delta}{\lambda^f}\begin{pmatrix}\theta & 0\end{pmatrix}\sum_{i=0}^{\infty}\delta^i \Phi^i\begin{pmatrix}c_t \\ c_{t-1}\end{pmatrix} + \frac{\delta}{\lambda^f}\varepsilon_{\pi,t}$$

$$= \frac{\delta}{\lambda^f}\begin{pmatrix}\theta & 0\end{pmatrix}\left(I_2-\delta\Phi\right)^{-1}\begin{pmatrix}c_t \\ c_{t-1}\end{pmatrix} + (1+\delta\xi)\,\varepsilon_{\pi,t}$$

$$= \frac{\delta}{\lambda^f}\begin{pmatrix}\theta & 0\end{pmatrix}\frac{1}{1-\delta\phi_1-\delta^2\phi_2}\begin{pmatrix}1 & \delta\phi_2 \\ \delta & 1-\delta\phi_1\end{pmatrix}\begin{pmatrix}c_t \\ c_{t-1}\end{pmatrix} + \tilde{\varepsilon}_{\pi,t}$$

$$= \frac{\delta}{\lambda^f}\frac{\theta}{1-\delta\phi_1-\delta^2\phi_2}\begin{pmatrix}1 & \delta\phi_2\end{pmatrix}\begin{pmatrix}c_t \\ c_{t-1}\end{pmatrix} + \tilde{\varepsilon}_{\pi,t}$$

where $\tilde{\varepsilon}_{\pi,t} := (1+\eta\xi)\,\varepsilon_{\pi,t}$. Finally, defining $\tilde{\theta} := {}^{\theta(1+\delta\xi)}/_{(1-\delta\phi_1-\delta^2\phi_2)}$, we get that

$$\pi_t = \xi\pi_{t-1} + \tilde{\theta}c_t + \tilde{\theta}\delta\phi_2 c_{t-1} + \tilde{\varepsilon}_{\pi,t}.$$

If Equation (18) had an intercept $\mu$, we get the additional term

$$\frac{\delta}{\lambda^f}\frac{\mu}{1-\delta} = \mu\frac{1+\xi\delta}{1-\delta}.$$

If the cycle was instead an AR(1) process, we obtain

$$\pi_t = \xi\pi_{t-1} + \tilde{\theta}c_t + \tilde{\varepsilon}_{\pi,t},$$

where $\tilde{\theta} = {}^{1}/_{(1-\phi_1\delta)}$. In the purely forward looking case, we have $\xi = 0$ and thus the autoregressive term drops.

## A.2 Optimization constraints

In the following, let $\Theta \subset \mathbb{R}^k, k \in \mathbb{Z}$ be the constrained parameter space. Moreover, let $\theta \in \Theta$ and $\psi \in \mathbb{R}^k$ denote the constrained and unconstrained parameter, respectively. To apply the delta method, for each constraint, we define a transformation function

$$g : \mathbb{R}^k \to \Theta, \quad g(\psi) = \theta.$$

For box constraints $\Theta = [a, b]$, we employ

$$g_{[a,b]}(\psi) = a + (b - a)\frac{1}{1 + e^{-\psi}} \qquad Dg_{[a,b]}(\psi) = (b - a)\frac{1}{1 + e^{-\psi}}\left(1 - \frac{1}{1 + e^{-\psi}}\right)$$

as $\lim_{\psi \to -\infty} g_{[a,b]}(\psi) = a$ and $\lim_{\psi \to \infty} g_{[a,b]}(\psi) = b$.

For a stationary AR($p$) process, the roots of the characteristic polynomial need to lie outside the unit circle. For $p = 1$, this implies that $\Theta = (-1, 1)$ and thus we employ box constraints with $a = -1 + \varepsilon, b = 1 - \varepsilon$, and $\varepsilon = 0.01$. For $p = 2$, the parameter space is given by $\Theta = \{\theta \in \mathbb{R}^2 : \theta_2 - \theta_1 < 1, \theta_2 + \theta_1 < 1, |\theta_2| < 1\}$ for which we have

$$g(\psi) = \begin{pmatrix} \frac{\psi_1}{1+|\psi_1|} + \frac{\psi_2}{1+|\psi_2|} \\ \\ -\frac{\psi_1}{1+|\psi_1|}\frac{\psi_2}{1+|\psi_2|} \end{pmatrix},$$

$$Dg(\psi) = \begin{pmatrix} \frac{-|\psi_1|}{(1+\psi_1)^2} + \frac{1}{1+|\psi_1|} & \frac{-|\psi_2|}{(1+\psi_2)^2} + \frac{1}{1+|\psi_2|} \\ \\ -\frac{\psi_2}{1+|\psi_2|}\left(\frac{-|\psi_1|}{(1+\psi_1)^2} + \frac{1}{1+|\psi_1|}\right) & -\frac{\psi_1}{1+|\psi_1|}\left(\frac{-|\psi_2|}{(1+\psi_2)^2} + \frac{1}{1+|\psi_2|}\right) \end{pmatrix}.$$

Note that for complex conjugate eigenvalues of the companion matrix it holds that $|\lambda| < {}^1\!/\!_4\left(\frac{\psi_1}{1+|\psi_1|} + \frac{\psi_2}{1+|\psi_2|}\right) < 1$.[15] For real eigenvalues, $|\lambda_1| = \frac{|\psi_1|}{1+|\psi_1|} < 1$ and $|\lambda_2| = \frac{|\psi_2|}{1+|\psi_2|} < 1$, and thus, the stationary criteria are fulfilled in either case. Recall the re-parameterized AR(2), i.e., $c_t = \phi_{c,1}c_{t-1} + \phi_{c,2}c_{t-2} + \varepsilon_{c,t}$ with

$$\phi_{c,1} = 2A\cos\left({}^{2\pi}\!/\!_{\tau}\right),$$

$$\phi_{c,2} = -A^2.$$

The complex conjugate roots $z_1, z_2$ of the characteristic polynomial $\Phi(z) = 1 - 2A\cos\left({}^{2\pi}\!/\!_{\tau}\right)z + A^2z^2$ are given by

$$z_{1/2} = A^{-1}\left(\cos\left({}^{2\pi}\!/\!_{\tau}\right) \pm i\sqrt{1 - \cos^2\left({}^{2\pi}\!/\!_{\tau}\right)}\right)$$

[15] If the eigenvalues of an AR(2) process are complex, they are in fact complex conjugates (Hamilton, 1994, Chapter 1).

and thus

$$\left|z_{1/2}\right| = A^{-1}\sqrt{\cos^2\left({}^{2\pi}/_{\tau}\right) + \sin^2\left({}^{2\pi}/_{\tau}\right)} = A^{-1}.$$

Hence, for stationarity we need that $|A| < 1$. Since the cosine is periodic, the first autoregressive coefficient $2A\cos\left({}^{2\pi}/_{\tau}\right)$ is not unique for $\tau \in \mathbb{R}$. For computational ease, we therefore restrict $\tau$ to lie in the interval $(2,\infty)$ which implies that $\cos\left({}^{2\pi}/_{\tau}\right)$ is strictly monotonically decreasing and that $\cos\left({}^{2\pi}/_{\tau}\right) \in (-1, 1)$. Note that if ${}^{2\pi}/_{\tau} \in \pi\mathbb{Z}$, the roots were in fact real. Define $\theta = (\phi_{c,1}, \phi_{c,2})$. Finally, we have

$$g(\psi) = \begin{pmatrix} 2g_{(-1,1)}(\psi_2)\cos\left(\frac{2\pi}{g_{(2,\infty)}(\psi_1)}\right) \\ -\left(g_{(-1,1)}(\psi_2)\right)^2 \end{pmatrix}$$

and

$$Dg(\psi) = \begin{pmatrix} -4\pi\frac{g_{(-1,1)}(\psi_2)Dg_{(2,\infty)}(\psi_1)}{\left(g_{(2,\infty)}(\psi_1)\right)^2}\sin\left(\frac{2\pi}{g_{(2,\infty)}(\psi_1)}\right) & 2Dg_{(-1,1)}(\psi_2)\cos\left(\frac{2\pi}{g_{(2,\infty)}(\psi_1)}\right) \\ 0 & -2g_{(-1,1)}(\psi_2)\,Dg_{(-1,1)}(\psi_1) \end{pmatrix}.$$

## A.3 Constraints in Bayesian analysis

We use two different techniques to draw samples from the constrained posterior distribution. The simplest way of enacting parameter constraints is by rejecting draws that do not meet the criteria. We use this procedure for draws from the multivariate Normal distribution and to meet stationarity criteria if applicable. For draws from the Normal distribution, the Beta distribution and the Inverse-Gamma distribution, we directly draw from the constrained distribution. Let $F$ be the posterior distribution of some parameter $\theta$ and let the parameter space be given by $\Theta = [a, b]$. Let $u$ be a draw from the standard Uniform distribution $\mathcal{U}(0, 1)$. Then, $\theta$ drawn from $F^{-1}\left(F(a) + u\left(F(b) - F(a)\right)\right)$ is a draw from the constrained full conditional (Gelfand et al., 1992).

Note that for the Inverse-Gamma distribution, we have that $\theta = {}^1/_x \sim \mathcal{IG}(\alpha, \beta)$ where $x \sim \mathcal{G}\left(\alpha, {}^1/_\beta\right)$ from which drawing is more convenient. Moreover, $\theta \in \Theta = [a, b]$ is equivalent to $x \in \left[{}^1/_b, {}^1/_a\right]$ and thus

$$\theta = \frac{1}{x}, x \sim F^{-1}\left(F\left({}^1/_b\right) + u\left(F\left({}^1/_a\right) - F\left({}^1/_b\right)\right)\right)$$

gives a draw from the constrained Inverse-Gamma distribution.

## A.4 Anchor recursions

Recall the definitions

$$a_{t+1} = \mathbb{E}\left[\alpha_{t+1}|Y_t\right], \qquad P_{t+1} = \text{var}\left(\alpha_{t+1}|Y_t\right),$$

$$a_{t|t} = \mathbb{E}[\alpha_t|Y_t], \qquad P_{t|t} = \text{var}(\alpha_t|Y_t),$$

$$\hat{\alpha}_t = \mathbb{E}\left[\alpha_t|Y_n\right], \qquad V_t = \text{var}\left(\alpha_t|Y_n\right),$$

the filtering recursions for $t = 1, \ldots, n$,

$$a_t = Ta_{t-1|t-1} + c, \qquad P_t = TP_{t-1|t-1}T' + RQR',$$

$$v_t = y_t - Za_t - Dz_t, \qquad F_t = ZP_tZ' + H,$$

$$K_t = P_tZ'F_t^{-1},$$

$$a_{t|t} = a_t + K_tv_t, \qquad P_{t|t} = P_t - K_tZP_t,$$

and the backwards smoothing recursion for $t = n, \ldots, 1$,

$$\hat{\alpha}_t = a_{t|t} + P_{t|t}T'P_{t+1}^{-1}(\hat{\alpha}_{t+1} - a_{t+1}),$$

$$V_t = P_{t|t} + P_{t|t}T'P_{t+1}^{-1}(V_{t+1} - P_{t+1})P_{t+1}^{-1}TP_{t|t}.$$

To compute the anchored values of a smoothed state component $\hat{\alpha}_{i,t} = r\hat{\alpha}_t$, where $r$ is a $1 \times m$ selection vector, we perform a backward and a forward recursion. By $\tilde{T}_k$ we denote the matrix exponential $T^k$. Furthermore, we define $C_k = \text{var}\left(\alpha_{n+k}|Y_n\right)$ and by $G_k = \sum_{i=0}^{k} T^i R_n Q R_n' T^{i'}$ we denote the innovation part of the conditional variance $C_k$. We first rely on the auxiliary recursion for $k = 1, \ldots, h$,

$$\tilde{T}_k = T\tilde{T}_{k-1},$$

$$G_k = G_n + TG_{k-1}T',$$

$$C_k = r\left(\tilde{T}_kP_{n|n}\tilde{T}_k' + G_k\right)r',$$

where we use the initialization $G_0 = R_n Q R_n', C_0 = P_{n|n}$ and $\tilde{T}_0 = T^0 = I$. Let $P_{t,n|n} = \operatorname{cov}(\alpha_t, \alpha_n | Y_n)$ with $P_{n,n|n} = P_{n|n}$. Then, the backward recursion for $t = n-1, \ldots, d$ is given by

$$\begin{aligned}
A_t &= P_{t|t} T' P_{t+1}^{-1}, \\
P_{t,n|n} &= A_t P_{t+1,n|n}, \\
w_t &= \left( r P_{t,n|n} \tilde{T}_h r' \right) / \left( r C_h r' \right), \\
\hat{\alpha}_{i,t}^a &= r \hat{\alpha}_t + w_t \left( s_{n+h} - r \tilde{T}_h \hat{\alpha}_n \right).
\end{aligned}$$

For the diffuse phase $t = d-1, \ldots, 1$, the first equality changes to

$$A_t = P_{t|t} T_t' \left( P_{t+1} + P_{\infty,t+1} \right)^{-1},$$

where $P_{\infty,t+1}$ contains the variance covariance matrix of the diffuse states, with zeros elsewhere. Last, the forward recursion for $t = n, \ldots, n+h$ is given by

$$\begin{aligned}
w_t &= \left( r C_{n-t} \tilde{T}_{n-t} r' \right) / \left( r C_h r' \right), \\
\hat{\alpha}_{i,t}^a &= r \hat{\alpha}_n \tilde{T}_{t-n} + w_t \left( s_{n+h} - r \tilde{T}_h \hat{\alpha}_n \right).
\end{aligned}$$

If the state vector contains a constant, the respective row and column is removed from all system matrices to ensure invertibility.

## A.5 Posterior distributions for Bayesian analysis

### A.5.1 Posterior of $p(\theta_c | c^n)$

We start by looking at the second term in Equation (17) involving the cycle. Let now $\phi = (\phi_1, \ldots, \phi_p)'$ and $c_t = \sum_{i=1}^{p} \phi_i c_{t-i} + \varepsilon_{c,t}$. Note that this model specification includes all three cycle models, namely an AR(1) or AR(2) process and a re-parametrized AR(2). It holds that

$$p\left(\phi, \sigma_c^2 \middle| c^n\right) = p\left(\phi \middle| \sigma_c^2, c^n\right) p\left(\sigma_c^2 \middle| \phi, c^n\right).$$

For each of the three models, a Gibbs step is implemented to draw the second term on the right hand side, $p(\sigma_c^2|\phi, c^n)$. We have that

$$p\left(\sigma_c^2\middle|\phi, c^n\right) \propto p\left(c_1, \ldots, c_p\middle|\phi, \sigma_c^2\right) \prod_{t=p+1}^{n} p\left(c_t\middle|\phi, \sigma_c^2, c^{t-1}\right) p\left(\sigma_c^2\right),$$

where

$$c_1, \ldots, c_p\middle|\phi, \sigma_c^2 \sim \mathcal{N}\left(0, \Sigma_{\phi,\sigma_c^2}\right),$$
$$\prod_{t=p+1}^{n} c_t\middle|\phi, \sigma_c^2, c^{t-1} \sim \mathcal{N}\left(\sum_{t=p+1}^{n}\sum_{i=1}^{p}\phi_i c_{t-i}, \sigma_c^2\right)$$

and $\Sigma_{\phi,\sigma_c^2}$ denotes the variance covariance matrix of $(c_1, \ldots, c_p)$ conditional on $\phi$ and $\sigma_c^2$. By conjugacy, we have that

$$p\left(\sigma_c^2\middle|\phi, c^n\right) = \mathcal{IG}\left(s_{c*}, \nu_{c*}\right)$$

with

$$\nu_{c*} = \nu_{c0} + n,$$
$$s_{c*} = s_{c0} + (c_1, \ldots, c_p)\left(\Sigma_{\phi,\sigma_c^2}\sigma_c^{-2}\right)^{-1}(c_1, \ldots, c_p)' + \sum_{t=1+p}^{n}\left(c_t - \sum_{i=1}^{p}\phi_i c_{t-i}\right)^2.$$

In contrast, sampling $\phi$ varies between the three models. For the AR(1) cycle, an analytical solution is available. Similar to above, it holds that

$$p\left(\phi\middle|\sigma_c^2, c^n\right) \propto p\left(c_1\middle|\phi, \sigma_c^2\right)\prod_{t=2}^{n} p\left(c_t\middle|\phi, \sigma_c^2, c^{t-1}\right) p(\phi). \tag{21}$$

Given that all three conditionals in Equation (21) are normal, it can be easily deduced that

$$p\left(\phi\middle|\sigma_c^2, c^n\right) = \mathcal{N}\left(\phi_*, V_{\phi*}\right)$$

with

$$V_{\phi_*} = \left(\frac{\sum_{t=2}^{n} c_{t-1}^2 - c_1^2}{\sigma_c^2} + \frac{1}{V_{\phi 0}}\right)^{-1},$$
$$\phi_* = V_{\phi*}\left(\frac{\sum_{t=2}^{n} c_t c_{t-1}}{\sigma_c^2} + \frac{\phi_0}{V_{\phi 0}}\right).$$

Summing up, for the AR(1) cycle specification, we draw sequentially from

$$\phi^k\middle|\sigma_c^{2\,k-1} \sim \mathcal{N}\left(\phi_*, V_{\phi*}\right) I_\phi,$$

$$\sigma_c^{2k}\Big|\phi^k \sim \mathcal{IG}\left(s_{c*}, \nu_{c*}\right) I_{\sigma_c^2},$$

where the superscript $k$ denotes the $k$-th draw.

If the cycle is instead specified to be a stationary AR(2) process with autoregressive parameters $\phi_1, \phi_2$, we can partially make use of the Normal conjugate framework. We assume the prior $p\left(\phi\right) = p\left(\phi_1, \phi_2\right) = \mathcal{N}\left(\mu_0, V_{\mu 0}\right), \mu_0 \in \Theta \subset \mathbb{R}^2,\ V_{\mu 0} \in \mathbb{R}^2_+$, where $\Theta$ indicates the stationary region. Note that we cannot apply the NIG conjugate framework since that would require known starting values $c_{-1}, c_0$. Instead, to obtain draws from $\phi_1, \phi_2|\sigma_c^2, c^n$, we can implement a Metropolis-Hastings step with proposal density $g$ given by

$$\prod_{t=3}^{T} p\left(c_t\Big|\phi_1, \phi_2, \sigma_c^2, c^{t-1}\right) p\left(\phi_1, \phi_2\right) = \mathcal{N}\left(\mu_*, V_{\mu*}\right) I_\phi$$

with

$$\begin{aligned} V_{\mu*} &= \left(\frac{X'X}{\sigma_c^2} + V_{\mu 0}^{-1}\right)^{-1}, \\ \mu_* &= V_{\mu 0}\left(\frac{X'y}{\sigma_c^2} + V_{\mu 0}^{-1}\mu_0\right), \end{aligned}$$

as implied by conjugacy with $y := \left(c_3, \ldots, c_n\right)', X := \Big(\left(c_2, \ldots, c_{n-1}\right)', \left(c_1, \ldots, c_{n-2}\right)'\Big)$. Given the target density $P\left(\phi\right) = p\left(\phi|\sigma_c^2, c^n\right)$, the acceptance probability simplifies to

$$\begin{aligned} \alpha &= \min\left\{1, \frac{P(\phi^k) g\left(\phi^{k-1}\Big|\phi^k\right)}{P(\phi^{k-1}) g\left(\phi^k|\phi^{k-1}\right)}\right\} \\ &= \min\left\{1, \frac{p\left(c_1, c_2\Big|\phi^k, \sigma_c^2\right) g\left(\phi^k\Big|\phi^{k-1}\right) g\left(\phi^{k-1}\Big|\phi^k\right)}{p\left(c_1, c_2|\phi^{k-1}, \sigma_c^2\right) g\left(\phi^{k-1}|\phi^k\right) g\left(\phi^k|\phi^{k-1}\right)}\right\} \\ &= \min\left\{1, \frac{p\left(c_1, c_2\Big|\phi^k, \sigma_c^2\right)}{p\left(c_1, c_2\Big|\phi_1^{k-1}, \sigma_c^2\right)}\right\} \\ &= \min\left\{1, \frac{\left|\Sigma_{\phi^{k-1}, \sigma_c^2}\right|^{1/2}}{\left|\Sigma_{\phi^k, \sigma_c^2}\right|^{1/2}} \exp\left\{\frac{1}{2}\left(c_1, c_2\right)\left(\Sigma_{\phi^{k-1}, \sigma_c^2}^{-1} - \Sigma_{\phi^k, \sigma_c^2}^{-1}\right)\left(c_1, c_2\right)'\right\}\right\} \end{aligned}$$

where $\Sigma_{\phi, \sigma_c^2}$ denotes the $2 \times 2$ covariance matrix of $(c_1, c_2)$ as a function of the involved parameters $\phi_1, \phi_2, \sigma_c^2$. Hence, for the AR(2) cycle specification, we draw sequentially from

$$\phi^k = \begin{cases} \tilde{\phi} & \text{if } u \leq \alpha \\ \phi^{k-1} & \text{if } u \leq \alpha \end{cases} \qquad \begin{matrix} \phi^k \Big| \sigma_c^{2^{k-1}} \sim \mathcal{N}\left(\mu_*, V_{\mu *}\right) I_\phi, \\ u \sim U[0,1], \end{matrix}$$

$$\sigma_c^{2^k} \Big| \phi^k \sim \mathcal{IG}\left(s_{c*}, \nu_{c*}\right) I_{\sigma_c^2}.$$

Last, we elaborate on the re-parametrized version of the cycle equation. We have that

$$p\left(A, \tau, \sigma_c^2 \Big| c^n\right) = p\left(A \Big| \tau, \sigma_c^2, c^n\right) p\left(\tau \Big| A, \sigma_c^2, c^n\right) p\left(\sigma_c^2 \Big| A, \tau, c^n\right).$$

For the third term on the right-hand-side, we can draw analogously to above with $\phi_1 = 2A\cos\left({}^{2\pi}/_{\tau}\right)$ and $\phi_2 = -A^2$. For the remaining two parameters $A$ and $\tau$ it holds that

$$p\left(A \Big| \tau, \sigma_c^2, c^n\right) \propto p\left(c_1, c_2 \Big| A, \tau, \sigma_c^2\right) \prod_{t=3}^{T} p\left(c_t \Big| A, \tau, \sigma_c^2, c^{t-1}\right) p\left(A\right),$$
$$p\left(\tau \Big| A, \sigma_c^2, c^n\right) \propto p\left(c_1, c_2 \Big| A, \tau, \sigma_c^2\right) \prod_{t=3}^{T} p\left(c_t \Big| A, \tau, \sigma_c^2, c^{t-1}\right) p\left(\tau\right).$$

Since the prior of $A$ and $\tau$ is a Beta distribution and the starting values $c_1, c_2$ are unknown, we cannot sample directly from these conditionals. However, in each case, all three involved densities are readily available. Havik et al. (2014) use the adaptive rejection Metropolis scheme (ARMS) introduced by Gilks et al. (1995), which we also adopt here. ARMS essentially combines adaptive rejection sampling (ARS) with an additional Metropolis-Hastings step in order to cope with possibly non-log-concave functions. If the target density is in fact log-concave, the Metropolis-Hastings step will always accept and ARMS falls back to ARS.

### A.5.2 Posterior of $p\left(\theta_p | p^n\right)$

For the posterior derivation of the trend equation, we start by considering the random walk with drift for which $\theta_p = \left(\omega, \sigma_\eta^2\right)$. The posterior can be obtained via a Gibbs step, i.e., we first sample from $\omega | \sigma_\eta^2$ and subsequently from $\sigma_\eta^2 | \omega$. We have that

$$p\left(\omega \Big| \sigma_\eta^2, p^n\right) = \prod_{t=2}^{n} p\left(\Delta p_t \Big| \omega, \sigma_\eta^2\right) p\left(\omega\right) \propto \mathcal{N}\left(\omega_*, V_{\omega *}\right)$$

with

$$V_{\omega*} = \left( \frac{n-1}{\sigma_\eta^2} + \frac{1}{V_{\omega 0}} \right)^{-1},$$
$$\omega_* = V_{\omega*} \left( \frac{\sum_{t=2}^{n} \Delta p_t}{\sigma_\eta^2} + \frac{\omega_0}{V_{\omega 0}} \right),$$

since $p\left(\Delta p_t \middle| \omega, \sigma_\eta^2\right) \propto \mathcal{N}\left(\omega, \sigma_\eta^2\right)$ and $p(\omega) = \mathcal{N}(\omega_0, V_{\omega 0})$. For the variance $\sigma_\eta^2$, we impose $p\left(\sigma_\eta^2\right) = \mathcal{IG}(s_{\eta 0}, \nu_{\eta 0})$ and thus we can deduce that

$$p\left(\sigma_\eta^2 \middle| \omega, p^n\right) = \prod_{t=2}^{n} p\left(\Delta p_t \middle| \omega, \sigma_\eta^2\right) p\left(\sigma_\eta^2\right) \propto \mathcal{IG}(s_{\eta*}, \nu_{\eta*})$$

with

$$\nu_{\eta*} = \nu_{\eta 0} + 1,$$
$$s_{\eta*} = s_{\eta 0} + \sum_{t=2}^{n} (\Delta p_t - \omega)^2.$$

In summary, for a random walk with constant drift, we draw from

$$\omega^k \middle| \sigma_\eta^{2k-1} \propto \mathcal{N}(\omega_*, V_{\omega*}) I_\omega,$$
$$\sigma_\eta^{2k} \middle| \omega^k \propto \mathcal{IG}(s_{\eta*}, \nu_{\eta*}) I_{\sigma_\eta^2}.$$

For a local linear trend model, the only parameter is $\theta_p = \sigma_\eta^2$ and thus no Gibbs step is necessary. Analogously to above, it holds that

$$p\left(\sigma_\eta^2 \middle| p^n\right) = \prod_{t=3}^{n} p\left(\Delta^2 p_t \middle| \sigma_\eta^2\right) p\left(\sigma_\eta^2\right) \propto \mathcal{IG}(s_{\eta*}, \nu_{\eta*})$$

with

$$\nu_{\eta*} = \nu_{\eta 0} + 1,$$
$$s_{\eta*} = s_{\eta 0} + \sum_{t=3}^{n} \left(\Delta^2 p_t\right)^2.$$

It remains to specify the posterior in the case of a damped trend with $\theta_p = \left(\omega, \phi_\eta, \sigma_\eta^2\right)$. Draws can be obtained using Gibbs and Metropolis-Hastings steps. Note that the NIG conjugate framework cannot be applied directly because we do not have given starting values for the autoregressive process, i.e., $\Delta p_1$ is not available. For the autoregressive parameter $\phi_\eta$, a Metropolis-Hastings step is implemented. For the target density, we have that

$$
\begin{aligned}
p\left(\phi_\eta \middle| \omega, \sigma_\eta^2, p^n\right) &\propto p\left(p^n \middle| \phi_\eta, \omega, \sigma_\eta^2\right) p\left(\phi_\eta\right) \\
&\propto p\left(\Delta p_2 \middle| \phi_\eta, \omega, \sigma_\eta^2\right) p\left(p_3, \ldots, p_n \middle| \phi_\eta, \omega, \sigma_\eta^2\right) p\left(\phi_\eta\right) \\
&\propto p\left(\Delta p_2 \middle| \phi_\eta, \omega, \sigma_\eta^2\right) \prod_{t=3}^{n-1} p\left(\Delta p_t \middle| \phi_\eta, \omega, \sigma_\eta^2, \Delta p_{t-1}\right) p\left(\phi_\eta\right). \qquad (22)
\end{aligned}
$$

The first term is readily given due to stationary of the trend drift and the second reflects the likelihood function of $\Delta p_t, t = 3, \ldots, n$. We have

$$
p\left(\Delta p_2 \middle| \phi_\eta, \omega, \sigma_\eta^2\right) = \mathcal{N}\left(\omega, \frac{\sigma_\eta^2}{\left(1-\phi_\eta^2\right)}\right), \qquad (23)
$$

$$
\prod_{t=3}^{n-1} p\left(\Delta p_t \middle| \phi_\eta, \omega, \sigma_\eta^2, \Delta p_{t-1}\right) = \mathcal{N}\left(\sum_{t=3}^{T} \left(1-\phi_\eta\right)\omega + \phi_\eta \Delta p_{t-1}, \sigma_\eta^2\right). \qquad (24)
$$

Hence, for the Metropolis-Hastings step, we can use the proposal distribution

$$
\prod_{t=3}^{n-1} p\left(\Delta p_t \middle| \phi_\eta, \omega, \sigma_\eta^2, \Delta p_{t-1}\right) p\left(\phi_\eta\right) \propto \mathcal{N}\left(\phi_{\eta *}, V_{\phi_\eta *}\right) I_{\phi_\eta},
$$

where

$$
\begin{aligned}
V_{\phi_\eta *} &= \left(\frac{\sum_{t=3}^{T} \left(\Delta p_{t-1} - \omega\right)^2}{\sigma_\eta^2} + \frac{1}{V_{\phi_\eta 0}}\right)^{-1}, \\
\phi_{\eta *} &= V_{\phi_\eta *}\left(\frac{\sum_{t=3}^{T} \left(\Delta p_t - \omega\right)\left(\Delta p_{t-1} - \omega\right)}{\sigma_\eta^2} + \frac{\phi_{\eta 0}}{V_{\phi_\eta 0}}\right)
\end{aligned}
$$

can be readily derived. Using similar arguments as above, the acceptance probability simplifies to

$$
\begin{aligned}
\alpha &= \min\left\{1, \frac{P(\phi_\eta^k) g\left(\phi_\eta^{k-1} \middle| \phi_\eta^k\right)}{P(\phi_\eta^{k-1}) g\left(\phi_\eta^k \middle| \phi_\eta^{k-1}\right)}\right\} \\
&= \min\left\{1, \frac{p\left(\Delta p_2 \middle| \phi_\eta^k, \omega, \sigma_\eta^2\right)}{p\left(\Delta p_2 \middle| \phi_\eta^{k-1}, \omega, \sigma_\eta^2\right)}\right\} \\
&= \min\left\{1, \exp\left\{\left(\phi_\eta^{2^k} - \phi_\eta^{2^{k-1}}\right) \frac{\left(\Delta p_2 - \omega\right)^2}{2\sigma_\eta^2}\right\}\right\}
\end{aligned}
$$

where the superscript $k$ again denotes the $k$-th draw and $P\left(\phi_\eta\right) = p\left(\phi_\eta \middle| \omega, \sigma_\eta^2, p^n\right)$ the target density. Recall that we assumed parameter independence, thus, using Equations (23) and (24) and due to conjugacy of the Normal distribution,

$$
p\left(\omega \middle| \phi_\eta, \sigma_\eta^2, p^n\right) = \mathcal{N}\left(\omega_*, V_{\omega *}\right),
$$

where

$$V_{\omega*} = \left( \frac{1-\phi_\eta^2 + (n-2)\left(1-\phi_\eta\right)^2}{\sigma_\eta^2} + \frac{1}{V_{\omega 0}} \right)^{-1},$$

$$\omega_* = V_{\omega*} \left( \frac{\left(1-\phi_\eta\right)\left(\Delta p_2 + \Delta p_n\right) + \left(1-\phi_\eta\right)^2 \sum_{t=3}^{n-1} \Delta p_t}{\sigma_\eta^2} + \frac{\omega_0}{V_{\omega 0}} \right).$$

Similarly, using the conjugacy of the Inverse-Gamma distribution and a similar factorization as in Equation (22), we have that

$$p\left(\sigma_\eta^2 \middle| \phi_\eta, \omega, p^n\right) = \mathrm{IG}\left(s_*, \nu_*\right),$$

where

$$\nu_* = \nu_0 + n - 1,$$

$$s_* = s_0 + \left(1-\phi_\eta^2\right)\left(\Delta p_2 - \omega\right)^2 + \sum_{t=3}^{T} \left(\Delta p_t - \omega\left(1-\phi_\eta^2\right) - \phi_\eta \Delta p_{t-1}\right)^2.$$

Summing up, to obtain a sample from the conditional $p\left(\omega, \phi_\eta, \sigma_\eta^2 \middle| p^n\right)$, we draw sequentially, i.e.,

$$\omega^k \middle| \phi_\eta^{k-1}, \sigma_\eta^{2^{k-1}}, p^n \sim \mathcal{N}\left(\omega_*, V_{\omega*}\right)$$

$$\phi_\eta^k = \begin{cases} \tilde{\phi}_\eta & \text{if } u \leq \alpha \\ \phi_\eta^{k-1} & \text{if } u > \alpha \end{cases} \qquad \tilde{\phi}_\eta \middle| \omega^k, \sigma_\eta^{2^{k-1}}, p^n \sim \mathcal{N}\left(\phi_{\eta*}, V_{\phi_{\eta*}}\right), \quad u \sim U[0,1],$$

$$\sigma_\eta^{2^k} \middle| \phi_\eta^k, \omega^k, p^n \sim \mathrm{IG}\left(s_*, \nu_*\right).$$

### A.5.3 Posterior of $p\left(\theta_{ind} | c^n, ind^n\right)$

For the third term in Equation (17), we start by considering the simplest case, for which the indicator equation does not involve cycle or autoregressive lags.[16] Thus, we can resort to conjugacy results of the Normal-Inverse-Gamma distribution. Let $x_t := (1, c_t)'$, $X := (x_1, \ldots, x_n)'$, $Y := (ind_1, \ldots, ind_n)'$ and $\beta := (\mu, \beta_{c,0})'$. It holds that

$$p\left(\beta, \sigma_{ind}^2 \middle| c^n, ind^n\right) = \mathrm{NIG}\left(\beta_*, Q_{\beta*}, s_{ind*}, \nu_{ind*}\right)$$

[16] All derivations that follow also hold when exogenous variables enter the indicator equation.

where

$$
\begin{aligned}
Q_* &= Q_0 + X'X, \\
\beta_* &= Q_*^{-1}\left(Q_0\beta_0 + X'Y\right), \\
\nu_* &= \nu_0 + T, \\
s_* &= s_0 + Y'Y + \beta_0' Q_0 \beta_0 - \beta_*' Q_* \beta_*
\end{aligned}
$$

(e.g. Fahrmeir et al., 2007). Hence, by the definition of the Normal-Inverse-Gamma distribution, we can draw sequentially from

$$
\begin{aligned}
\sigma_{ind}^2 &\sim \mathrm{IG}\left(s_*, \nu_*\right), \\
\beta|\sigma_{ind}^2 &\sim \mathcal{N}\left(\beta_*, \sigma_{ind}^2 Q_*\right).
\end{aligned}
$$

In the case that the error is specified as an AR(1) process, i.e., $\hat{\varepsilon}_t = \phi_{\hat{\varepsilon}}\hat{\varepsilon}_{t-1} + \varepsilon_{ind,t}$, we can factorize

$$
\mathrm{p}\left(\beta, \phi_{\hat{\varepsilon}}, \sigma_{ind}^2 \middle| c^n, ind^n\right) = p\left(\beta, \sigma_{ind}^2 \middle| \phi_{\hat{\varepsilon}}, c^n, ind^n\right) p\left(\phi_{\hat{\varepsilon}} \middle| \beta, \sigma_{ind}^2, c^n, ind^n\right)
$$

and sample from $p\left(\phi_{\hat{\varepsilon}}|\beta, \sigma_{ind}^2, c^n, ind^n\right)$ in a second Gibbs step. It holds that

$$
p\left(\phi_{\hat{\varepsilon}} \middle| \beta, \sigma_{ind}^2, c^n, ind^n\right) \propto p\left(\hat{\varepsilon}_1 \middle| \phi_{\hat{\varepsilon}}, \sigma_{ind}^2\right) \prod_{t=2}^{T} p\left(\hat{\varepsilon}_t \middle| \hat{\varepsilon}_{t-1}, \phi_{\hat{\varepsilon}}, \sigma_{ind}^2\right) p\left(\phi_{\hat{\varepsilon}}\right)
$$

and all three conditionals are normal. Thus, as for the autoregressive coefficient in the AR(1) cycle equation above, it can be easily deduced that

$$
p\left(\phi_{\hat{\varepsilon}} \middle| \beta, \sigma_{ind}^2, c^n, ind^n\right) = \mathcal{N}\left(\phi_{\hat{\varepsilon}*}, V_{\phi_{\hat{\varepsilon}}*}\right)
$$

with

$$
\begin{aligned}
V_{\phi_{\hat{\varepsilon}}*} &= \left(\frac{\sum_{t=2}^{T} \hat{\varepsilon}_{t-1}^2 - \hat{\varepsilon}_1^2}{\sigma_{ind}^2} + \frac{1}{V_{\phi_{\hat{\varepsilon}0}}}\right)^{-1}, \\
\phi_{\hat{\varepsilon}*} &= V_{\phi_{\hat{\varepsilon}}*}\left(\frac{\sum_{t=2}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-1}}{\sigma_{ind}^2} + \frac{\phi_{\hat{\varepsilon}0}}{V_{\phi_{\hat{\varepsilon}0}}}\right),
\end{aligned}
$$

and $\hat{\varepsilon} = ind_t - \mu - \beta_{c,0}c_t$. For an AR(2) error process, posterior draws are obtained in a Metropolis-Hastings step, analogously as for the AR(2) cycle above.

Last, we consider the most general case with autoregressive terms and cycle lags included in the indicator equation. To obtain a sample of the posterior, we implement a Metropolis-Hastings step. Let $ind_{t-1}, \ldots, ind_{t-\tilde{p}}$ and $c_t, \ldots, c_{t-p}, \tilde{p}, p \in \mathbb{Z}$ be included in the equation with coefficients $\phi_1, \ldots, \phi_{\tilde{p}}, \beta_{c,0}, \ldots, \beta_{c,p}$. Define $\beta = (\mu, \phi_1, \ldots, \phi_{\tilde{p}}, \beta_{c,0}, \ldots, \beta_{c,p})$ and $q = \max\{p, \tilde{p}\}$. The posterior with iid Normal error term factorizes, i.e.,

$$\begin{aligned}
&p\left(\beta, \sigma^2_{ind} \middle| \phi_{\hat{\varepsilon}}, c^n, ind^n\right) \\
&\quad \propto p\left(ind^q \middle| c^q, \beta, \sigma^2_{ind}\right) \prod_{t=q+1}^{n} p\left(ind_t \middle| c^n, ind^{n-1}, \beta, \sigma^2_{ind}\right) p\left(\beta, \sigma^2_{ind}\right) \\
&\quad \propto p\left(ind^q \middle| \beta, \sigma^2_{ind}\right) p\left(c^p\right) \prod_{t=q+1}^{n} p\left(ind_t \middle| c^n, ind^{n-1}, \beta, \sigma^2_{ind}\right) p\left(\beta, \sigma^2_{ind}\right).
\end{aligned}$$

NIG conjugacy can be applied to the second and third term on the right hand side and thus the product of two terms can be used as proposal distribution $g$. Let $P$ denote the target density. The acceptance probability simplifies to

$$\alpha = \min\left\{1, \frac{p\left(ind^q \middle| \beta^{k-1}, {\sigma^2_{ind}}^{k-1}\right)}{p\left(ind^q \middle| \beta^{k}, {\sigma^2_{ind}}^{k}\right)}\right\}.$$

It remains to specify mean and variance of the Normal vector $ind^q|\beta, \sigma^2_{ind}$. For all $t \in \mathbb{Z}$,

$$\begin{aligned}
\mu_{ind} &= \mathbb{E}\left[ind_t|\beta, \sigma^2_{ind}\right] = \frac{\mu}{1 - \sum_{i=1}^{\tilde{p}} \phi_i} \\
\Sigma_{ind^p} &:= \operatorname{cov}\left(ind^q|\beta, \sigma^2_{ind}\right).
\end{aligned}$$

For a derivation of $\Sigma_{ind^p}$, see Appendix A.8. Hence, the acceptance probability is given by

$$\alpha = \min\left\{1, \frac{\left|\Sigma^k_{ind^p}\right|^{-1/2} \exp\left\{-1/2\left(ind^p - \mu^k_{ind^p}\right)' {\Sigma^{-1}_{ind^p}}^{k}\left(ind^p - \mu^k_{ind^p}\right)\right\}}{\left|\Sigma^{k-1}_{ind^p}\right|^{-1/2} \exp\left\{-1/2\left(ind^p - \mu^{k-1}_{ind^p}\right)' {\Sigma^{-1}_{ind^p}}^{k-1}\left(ind^p - \mu^{k-1}_{ind^p}\right)\right\}}\right\}.$$

### A.5.4 Inverse-Gamma distribution

For some $x \sim \mathcal{IG}(s, \nu)$, where $s$ denotes the location and $\nu$ the degrees of freedom, the density function is given by

$$c_g(s, \nu)^{-1} x^{-\frac{1}{2}(\nu+2)} \exp\left\{-s/2x\right\}$$

where $c_g(s,\nu) := \Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\nu/2}$. Under this parametrization, we have that

$$\mathbb{E}[x] = \frac{s}{\nu-2}, \quad \text{var}(x) = \frac{s}{\nu-4}\left(\frac{s}{\nu-2}\right)^2,$$

which implies that

$$\nu = \frac{2\mathbb{E}[x]^2}{\text{var}(x)} + 4, \quad s = \mathbb{E}[x](\nu-2).$$

Note that if $\mathbb{E}[x] = \sqrt{\text{var}(x)}$ we have that $\nu = 6$.

Alternatively, it can be specified via shape $\alpha = \nu/2$ and scale $\beta = s/2$, i.e., $x \sim \mathcal{IG}(\alpha,\beta)$, with density function

$$\tilde{c}_g(\alpha,\beta)\, x^{-(\alpha+1)} \exp\{-\beta/x\}$$

where $\tilde{c}_g(\alpha,\beta) := \beta^\alpha/\Gamma(\alpha)$. Here, it holds that

$$\begin{aligned} \mathbb{E}[x] &= \frac{\beta}{\alpha-1} && \text{for } \alpha > 1, \\ \text{var}(x) &= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} && \text{for } \alpha > 2, \end{aligned}$$

from which it follows that

$$\alpha = \frac{\mathbb{E}[x]^2}{\text{var}(x)} + 2, \quad \beta = \mathbb{E}[x](\alpha-1).$$

Note that if $\mathbb{E}[x] = \sqrt{\text{var}(x)}$ we have that $\alpha = 3$.

## A.6 Variance restrictions

**RGAP** initializes variance restrictions using information obtained from the supplied data. For simplicity, the trend variance is set to zero in case of a damped trend or local linear trend specification. In those cases, the trend variation originates from the trend drift innovations. Thus, restrictions on three innovation variances are necessary.

The function `initializeVar` offers two methods to initialize the variance restrictions. If `type = "basic"` is specified, the upper bound of the trend and cycle variances is set to the sample variance of the differenced first observation series $\widehat{\text{var}}(\Delta y_1^n)$, where $y_i^n = (y_{i,1}, \ldots, y_{i,n})$ and $y_{i,t}$ denotes the $i$-th component of the observation vector at time

$t$. Similarly, the upper bound for the second equation innovation variance is set to the sample variance of the second equation series $\widehat{\mathrm{var}}\left(y_2^n\right)$. The lower bounds are set to zero.

If `type = "hp"`, the HP-filter is used to obtain a trend and cycle series of the first observation equation. For both series, estimates of the innovation variance are acquired through appropriate ARIMA models.[17] For the second observation equation, the innovation variance is obtained by OLS using the HP-cycle as explanatory variable. Given these estimated variances $\hat{\sigma}_i^2, i \in \{1, 2, 3\}$, the box constraints for the variance are set to the 80% confidence interval of an Inverse-Gamma distributed variable with mean and standard deviation $\hat{\sigma}_i$.

## A.7 Signal-to-noise ratio

One way to impose variability on the cycle and thereby smoothness on the trend when applying MLE is to put tight box constraints on the variance parameters. Alternatively, a specific smoothness can be achieved by fixing the signal-to-noise ratio $q := \frac{\sigma_p^2 + \sigma_\eta^2}{\sigma_c^2}$. **RGAP** allows to specify $q$ instead of estimating all variances separately. If $q$ is specified, $\sigma_p^2$ will be set to zero if both $\sigma_p^2$ and $\sigma_\eta^2$ are present in the model specification. During optimization, the cycle variance $\sigma_c^2$ is estimated while the (remaining) trend variance is set to $q\sigma_c^2$.

## A.8 Variance covariance matrix of *ind*

The indicator equation is given by $ind_t = \mu + \sum_{i=1}^{\tilde{p}} \phi_i ind_{t-i} + \sum_{j=0}^{p} \beta_j c_{t-i} + \varepsilon_t$ with $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_{ind}^2\right)$ and $c_t$ follows a stationary AR($p_c$) process. Let $q = \max\{p, \tilde{p}\}$. Define

$$\gamma(k) := \mathrm{cov}\left(ind_t, ind_{t-k}\right) = \mathbb{E}\left[\left(ind_t - \mu_{ind}\right)\left(ind_{t-k} - \mu_{ind}\right)\right] = \gamma(-k),$$

$$\tilde{\gamma}(k-\ell) := \mathrm{cov}\left(c_{t-\ell}, ind_{t-k}\right) = \mathbb{E}\left[c_{t-\ell}\left(ind_{t-k} - \mu_{ind}\right)\right].$$

[17] For the damped trend and for the cycle, suitable AR($p$) models are estimated. For a random walk with constant or stochastic drift, a suitable ARIMA($0, k, 0$) model is estimated.

The Yule-Walker equations are given by

$$\begin{aligned}
\gamma(0) &= \mathbb{E}\left[(ind_t - \mu_{ind})(ind_t - \mu_{ind})\right] \\
&= \mathbb{E}\left[\left(\mu + \sum_{i=1}^{\tilde{p}} \phi_i ind_{t-i} + \sum_{j=0}^{p} \beta_j c_{t-j} + \varepsilon_t - \mu_{ind}\right)(ind_t - \mu_{ind})\right] \\
&= \sum_{i=1}^{\tilde{p}} \phi_i \gamma(-i) + \sum_{j=0}^{p} \beta_j \tilde{\gamma}(-j), \\
\gamma(k) &= \sum_{i=1}^{\tilde{p}} \phi_i \gamma(k-i) + \sum_{j=0}^{p} \beta_j \tilde{\gamma}(k-j).
\end{aligned}$$

We have that $\Phi\Gamma_{0:q} = y$, where $\Gamma_{0:q} := (\gamma(0), \ldots, \gamma(q))'$ and

$$\Phi := \begin{pmatrix} \Phi_{1,\tilde{p}+1\times\tilde{p}+1} & 0_{\tilde{p}+1\times q-\tilde{p}} \\ 0_{q-\tilde{p}\times 1} & \Phi_{2,q-\tilde{p}\times q} \end{pmatrix}$$

with

$$\Phi_1 = \begin{pmatrix}
1 & -\phi_1 & -\phi_2 & \cdots & & & -\phi_{\tilde{p}} \\
-\phi_1 & 1-\phi_2 & -\phi_3 & \cdots & & -\phi_{\tilde{p}} & \\
-\phi_2 & -\phi_1-\phi_3 & 1-\phi_4 & \cdots & -\phi_{\tilde{p}} & & \\
\vdots & & & \ddots & & & \\
\vdots & & & & \ddots & & \\
-\phi_{\tilde{p}-1} & -\phi_{\tilde{p}}-\phi_{\tilde{p}-2} & -\phi_{\tilde{p}-3} & & -\phi_1 & 1 & 0 \\
-\phi_{\tilde{p}} & -\phi_{\tilde{p}-1} & -\phi_{\tilde{p}-2} & & & -\phi_1 & 1
\end{pmatrix},$$

$$\Phi_2 = \begin{pmatrix}
-\phi_{\tilde{p}} & -\phi_{\tilde{p}-1} & \cdots & -\phi_1 & 1 & & \\
& \ddots & & & & \ddots & \\
& & -\phi_{\tilde{p}} & -\phi_{\tilde{p}-1} & \cdots & -\phi_1 & 1
\end{pmatrix},$$

$$y = \begin{pmatrix} \sum_{j=0}^{p} \beta_j \tilde{\gamma}(-j) + \sigma_{ind}^2 & \sum_{j=0}^{p} \beta_j \tilde{\gamma}(1-j) & \cdots & \sum_{j=0}^{p} \beta_j \tilde{\gamma}(q-j) \end{pmatrix}',$$

which implies $\Gamma_{0:q} = \Phi^{-1}y$. It remains to specify the covariances between the indicator and cycle processes $\tilde{\gamma}(k)$ for $k = -p, \ldots, q$. As above, the covariances can be obtained using the Yule-Walker equations, i.e.,

$$\tilde{\gamma}(0) = \mathbb{E}\left[c_t (ind_t - \mu_{ind})\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[c_t\left(\mu + \sum_{i=1}^{\tilde{p}} \phi_i ind_{t-i} + \sum_{j=0}^{p} \beta_j c_{t-i} + \varepsilon_t - \mu_{ind}\right)\right] \\
&= \sum_{i=1}^{\tilde{p}} \phi_i \tilde{\gamma}(-i) + \sum_{j=0}^{p} \beta_j \gamma_c(-j), \\
\tilde{\gamma}(k) &= \sum_{i=1}^{\tilde{p}} \phi_i \tilde{\gamma}(i+k) + \sum_{j=0}^{p} \beta_j \gamma_c(j+k) = \sum_{j=1}^{p_c} \phi_{c,j} \tilde{\gamma}(k-j).
\end{aligned}
$$

The cycle covariance funtion $\gamma_c(k)$ is given by the parameters of the cycle process. We have that $\tilde{\Phi}\tilde{\Gamma}_{-p:q} = \tilde{y}$ where $\tilde{\Gamma}_{-p:q} := (\tilde{\gamma}(-p), \ldots, \tilde{\gamma}(q))'$ and

$$
\tilde{\Phi} := \begin{pmatrix} \tilde{\Phi}_{1,p+1\times p+1+\tilde{p}} & 0_{p+1\times q-\tilde{p}} \\ 0_{q\times p+1-p_c} & \tilde{\Phi}_{2,q\times p_c+q} \end{pmatrix}
$$

with

$$
\tilde{\Phi}_1 = \begin{pmatrix} 1 & -\phi_1 & -\phi_2 & \cdots & -\phi_{\tilde{p}} & & & & \\ & \ddots & & & & \ddots & & & \\ & & 1 & -\phi_1 & -\phi_2 & \cdots & -\phi_{\tilde{p}} \end{pmatrix},
$$

$$
\tilde{\Phi}_2 = \begin{pmatrix} -\phi_{c,p_c} & -\phi_{c,p_c-1} & \cdots & -\phi_{c,1} & 1 & & & & \\ & \ddots & & & & \ddots & & & \\ & & -\phi_{c,p_c} & -\phi_{c,p_c-1} & \cdots & -\phi_{c,1} & 1 \end{pmatrix},
$$

$$
\tilde{y} = \begin{pmatrix} \sum_{j=0}^{p} \beta_j \gamma_c(j-p) & \cdots & \sum_{j=0}^{p} \beta_j \gamma_c(j) & 0_{1\times q} \end{pmatrix}'.
$$

Note that we must have $1 - p_c \geq -p \Leftrightarrow p \geq p_c - 1$ to be able to retrieve all covariances. However, if this requirement is not fulfilled, we can simply extend $\tilde{\Phi}_1$. Finally, $\tilde{\Gamma}_{-p:q} = \tilde{\Phi}^{-1}\tilde{y}$.